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Abstract

Let
$$\mathcal{F}(\mathbb{R}^2) = \{f \in \mathbf{L}_\infty(\mathbb{R}^2) \cap \mathbf{L}_1(\mathbb{R}^2) : f \geq 0\}.$$
Suppose $s \in \mathcal{F}(\mathbb{R}^2)$ and $\gamma : \mathbb{R} \to [0, \infty)$. Suppose γ is zero at zero, positive away from zero and convex. For $f \in \mathcal{F}(\Omega)$ let

$$F(f) = \int_\Omega \gamma(f(x) - s(x))\, d\mathcal{L}^2 x;$$

$\mathcal{L}^2$ here is Lebesgue measure on $\mathbb{R}^2$. In the denoising literature F would be called a *fidelity* in that it measures how much f differs from s which could be a noisy grayscale image. Suppose $0 < \epsilon < \infty$ and let

$$\mathbf{n}_\epsilon^{loc}(F)$$

be the set of those $f \in \mathcal{F}(\mathbb{R}^2)$ such that $\mathbf{TV}(f) < \infty$ and

$$\epsilon \mathbf{TV}(f) + F(f) \leq \epsilon \mathbf{TV}(g) + F(g) \quad \text{for } g \in \mathbf{k}(f);$$

here $\mathbf{TV}(f)$ is the total variation of f and $\mathbf{k}(f)$ is the set of $g \in \mathcal{F}(\mathbb{R}^2)$ such that $g = f$ off some compact subset of $\mathbb{R}^2$. A member of $\mathbf{m}_\epsilon^{loc}(F)$ is called a *total variation regularization of s (with smoothing parameter ϵ).* Rudin, Osher and Fatemi in [ROF] and Chan and Esedoglu in [CE] have studied total variation regularizations of F where $\gamma(y) = y^2$ and $\gamma(y) = y$, $y \in \mathbb{R}$, respectively.

Our purpose in this paper is to determine $\mathbf{m}_\epsilon^{loc}(F)$ when s is the indicator function of a compact convex subset of $\mathbb{R}^2$.

While taking $s = 1_S$, S compact and convex, is certainly not representative of the functions s which occur in image denoising, we hope this result sheds some light on the nature of total variation regularizations. In addition, one can test computational schemes for total variation regularization against these examples. Examples where S is *not* convex will appear in a later paper.

Total variation regularization for image denoising; II. Examples.

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1 Introduction.

1.1 Total variation.

This work is based on the notion of the total variation of a locally summable function on $\mathbb{R}^2$, which we now define.

Definition 1.1. Suppose $f \in \mathbf{L}_1^{loc}(\mathbb{R}^2)$. We let

$$\mathbf{TV}(f) = \sup \left\{ \int f \,\mathbf{div}\, X \, d\mathcal{L}^2 : X \in \mathcal{X}(\mathbb{R}^2) \text{ and } |X| \leq 1 \right\}$$

and call this nonnegative extended real number the **total variation of** f; here $\mathcal{X}(\mathbb{R}^2)$ is the vector space of smooth compactly supported vector fields on $\mathbb{R}^2$ and and $\mathcal{L}^2$ is Lebesgue measure on $\mathbb{R}^2$.

In particular, if f is continuously differentiable on $\mathbb{R}^2$ then

$$\mathbf{TV}(f) = \int |\nabla f| \, d\mathcal{L}^2. \tag{1}$$

Moreover, if E a Lebesgue measurable subset of $\mathbb{R}^2$ with Lipschitz boundary then $\mathbf{TV}(E)$ equals the length of the boundary; *here and in what follows we frequently identify a subset E of $\mathbb{R}^2$ with its indicator function 1_E.*

1.2 Total variation regularization.

We let

$$\mathcal{F}(\mathbb{R}^2) = \{f \in \mathbf{L}_1(\mathbb{R}^2) \cap \mathbf{L}_\infty(\mathbb{R}^2) : f \geq 0\}.$$

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Definition 1.2. Suppose $F : \mathcal{F}(\mathbb{R}^2) \to \mathbb{R}$ and $0 < \epsilon < \infty$. We let

$$F_\epsilon(g) = \epsilon \mathbf{TV}(g) + F(g) \quad \text{for } g \in \mathcal{F}(\mathbb{R}^2)$$

and we let

$$\mathbf{m}_\epsilon^{loc}(F) = \{f \in \mathcal{F}(\mathbb{R}^2) : \mathbf{TV}(f) < \infty \text{ and } F_\epsilon(f) \leq F_\epsilon(g) \text{ for } g \in \mathbf{k}(f)\};$$

here $\mathbf{k}(f)$ is the set of $g \in \mathcal{F}(\mathbb{R}^2)$ such that, for some compact subset K of $\mathbb{R}^2$, $g(x) = f(x)$ for $\mathcal{L}^2$ almost all $x \in \mathbb{R}^2 \sim K$.

Suppose

(i) $s \in \mathcal{F}(\mathbb{R}^2)$;

(ii) $\gamma : \mathbb{R} \to [0, \infty)$, γ is convex, $\gamma(0) = 0$ and $\gamma(y) > 0$ if $y \in \mathbb{R} \sim \{0\}$;

(iii) $F(f) = \int_{\mathbb{R}^2} \gamma(f(x) - s(x))\, d\mathcal{L}^2 x$ for $f \in \mathcal{F}(\mathbb{R}^2)$.

Here s could be a grayscale representation of a degraded image which we wish to denoise. In the context of denoising F would be called a **fidelity**; it is a measure of how much f differs from s. If $0 < \epsilon < \infty$ the members of $\mathbf{m}_\epsilon(F)$ would be called **total variation regularizations of s (with respect to the fidelity F and smoothing parameter ϵ).**

In the literature F_ϵ is often replaced by F_ϵ/ϵ and $\lambda = 1/\epsilon$ is thought of as a Lagrange multiplier.

For a very informative discussion of the use of total variation regularizations in the field of image processing see the Introduction of [CE]. We will not discuss image processing any further except to note that the notion of total variation regularization in image processing is useful for other purposes besides denoising.

For the remainder of this paper let ϵ be a positive real number, let γ be as in (ii) above, let S be a compact convex subset of $\mathbb{R}^2$ such the $\mathcal{L}^2(S) > 0$ and let $s = 1_S$. Our main purpose in this paper is to determine $\mathbf{m}_\epsilon^{loc}(F)$.

While taking $s = 1_S$, S compact and convex, is certainly not representative of the functions s which occur in image denoising, we hope this result sheds some light on the nature of total variation regularizations. In addition, one can test computational schemes for total variation regularization against these examples. Examples where S is *not* convex will appear in [AW2]

1.3 The main theorem.

For reasons which will become clear shortly we introduce the following terminology.

Let

$$\beta(y) = \limsup_{z \downarrow y} \frac{\gamma(z) - \gamma(y)}{z - y} \quad \text{for } y \in \mathbb{R}.$$

Note that β is nonincreasing and negative on $(-\infty, 0)$ and nondecreasing and positive on $(0, \infty)$.

1.4 The sets T_r, $0 < r < \infty$, and the function Φ.

Definition 1.3. Whenever $a \in \mathbb{R}^2$ and $0 < r < \infty$ we let

$$\mathbf{U}(a, r) = \{x \in \mathbb{R}^2 : |x - a| < r\} \quad \text{we let} \quad \mathbf{B}(a, r) = \{x \in \mathbb{R}^2 : |x - a| \leq r\}.$$

Definition 1.4. Suppose $0 < r < \infty$. Let

$$C_r = \{c \in S : \{x \in \mathbb{R}^2 : \mathbf{B}(c, r) \subset S\}$$

and let

$$T_r = \cup\{\mathbf{B}(c, r) : c \in C_r\} = \{x \in \mathbb{R}^2 : \mathbf{dist}\,(x, C_r) \leq r\}.$$

Evidently,

$$0 < r \leq s < \infty \;\Leftrightarrow\; T_s \subset T_r;$$

Let

$$R = \sup\{r \in (0, \infty) : T_r \neq \emptyset\}.$$

Since S is compact with nonempty interior it follows that $0 < R < \infty$ and

$$T_r \neq \emptyset \;\Leftrightarrow\; 0 < r \leq R.$$

Suppose $0 < r < \infty$ and b is a boundary point of T_r which is interior to S; among other things, we will show below that there are an open neighborhood G of b and $c \in C_r$ such

$$T_r \cap G = \mathbf{B}(c, r) \cap G.$$

Let

$$\Phi(r) = \mathbf{TV}(T_r) - \frac{\mathcal{L}^2(T_r)}{r} \quad \text{for } 0 < r \leq R.$$

We will prove various facts about T_r, $0 < r \leq R$, and Φ in Section 4 below, among which is the following.

Proposition 1.1. Φ is increasing and continuous.

Remark 1.1. In fact, as the reader may want to try to show after reading the proof of Proposition 1.1, Φ is Lipschitzian on any interval $(0, r)$, $0 < r < R$.

1.5 The functions η and Ψ.

For each $y \in (0, 1)$ let

$$\eta(y) = -\frac{\epsilon}{\beta(y-1)} \quad \text{and let} \quad W_y = T_{\eta(y)}.$$

Note that η is nondecreasing.

For example, if $\gamma(y) = |y|$, $y \in \mathbb{R}$, then

$$\beta(y) = \begin{cases} -1 & \text{if } -\infty < y < 0, \\ 1 & \text{if } 0 \leq y < \infty \end{cases} \quad \text{so} \quad \eta(y) = \epsilon \quad \text{for } 0 < y < 1$$

and if $\gamma(y) = y^2/2$, $y \in \mathbb{R}$, then

$$\beta(y) = y \quad \text{so} \quad \eta(y) = \frac{\epsilon}{1-y} \quad \text{for } 0 < y < 1.$$

Let

$$I = \{y \in (0,1) : 0 < \eta(y) \leq R\} = \{y \in (0,1) : W_y \neq \emptyset\}.$$

Note that I is an interval, possibly empty. Let

$$\Psi(y) = \epsilon \mathbf{TV}(W_y) + \beta(y-1)\mathcal{L}^2(W_y) = \epsilon\Phi(\eta(y)) \quad \text{for } y \in I;$$

since η and Φ are nondecreasing it follows that Ψ is nondecreasing Let

$$I_- = \{y \in I : \Psi(y) < 0\}, \quad I_0 = \{y \in I : \Psi(y) = 0\}, \quad I_+ = \{y \in I : \Psi(y) > 0\};$$

since Ψ is nonincreasing we find that I_-, I_0, I_+ are intervals, that $y_- < y_0$ if $y_- \in I_-$ and $y_0 \in I_0$ and that $y_0 < y_+$ if $y_0 \in I_0$ and $y_+ \in I_+$. Note that if γ is strictly convex then Ψ is increasing in which case I_0 contains at most one point.

1.6 Statement of the Main Theorem.

Let

$$J = \begin{cases} \{0\} & \text{if } I_- \cup I_0 = \emptyset; \\ \{\sup I_-\} & \text{if } I_- \neq \emptyset \text{ and } I_0 = \emptyset; \\ [\inf I_0, \sup I_0] & \text{if } I_0 \neq \emptyset. \end{cases}$$

Our purpose in this paper is to prove the following Theorem.

Theorem 1.1. $f \in \mathbf{m}_\epsilon^{loc}(F)$ if and only for some $Y \in J$ we have

$$f(x) = \mathcal{L}^1([0,Y] \cap \{y \in (0,1) : x \in W_y\}) \quad \text{for } \mathcal{L}^2 \text{ almost all } x. \tag{2}$$

See [AW1, 1.11] for the case when S is a square and, for some $p \in [1, \infty)$, $\gamma(y) = |y|^p/p$, $y \in \mathbb{R}$.

1.7 Acknowledgments.

It is a pleasure to acknowledge useful conversations with Kevin Vixie and Selim Esedoglu.

2 Functionals on sets.

It will be useful to extend the foregoing notions to functionals defined on sets, as follows.

We let

$$\mathcal{M}(\mathbb{R}^2) = \left\{D : D \subset \mathbb{R}^2 \text{ and } 1_D \in \mathcal{F}(\mathbb{R}^2)\right\};$$

thus a subset D of $\mathbb{R}^2$ belongs to $\mathcal{M}(\mathbb{R}^2)$ if and only if D is Lebesgue measurable and $\mathcal{L}^n(D) < \infty$.

Suppose $M : \mathcal{M}(\mathbb{R}^2) \to \mathbb{R}$ and $0 < \zeta < \infty$. We let

$$M_\zeta(E) = \zeta \mathbf{TV}(E) + M(E) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2)$$

and we let

$$\mathbf{n}_\zeta^{loc}(M)$$

be the family of those $D \in \mathcal{M}(\mathbb{R}^2)$ such that $\mathbf{TV}(D) < \infty$ and $M_\zeta(D) \leq M_\zeta(E)$ whenever $E \in \mathcal{M}(\Omega)$ and $1_E \in \mathbf{k}(1_D)$.

Let

$$\Sigma(D, E) = \mathcal{L}^2((D \sim E) \cup (E \sim D)) = \int |1_D - 1_E| \, d\mathcal{L}^2$$

whenever D, E are Lebesgue measurable subsets of $\mathbb{R}^2$.

Now fix $X \in \mathcal{M}(\mathbb{R}^2)$ and let

$$M_X(E) = \Sigma(X, E) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).$$

In [CE] Chan and Esedoglu study

$$\mathbf{n}_\zeta^{loc}(M_X);$$

their work was the starting point for the results of [AW1], this paper and [AW2].

Let

$$V_X(E) = -\mathcal{L}^2(E \cap X) + \mathcal{L}^2(E \sim X) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).$$

Since $M_X(E) = V_X(E) - \mathcal{L}^2(X)$ for $E \in \mathcal{M}(\mathbb{R}^2)$ we find that

$$\mathbf{n}_\zeta^{loc}(M_X) = \mathbf{n}_\zeta^{loc}(V_X).$$

Since $V_X(\emptyset) = 0$ we find it easier to work with V_X than with M_X.

2.1 Characterization of $\mathbf{n}_\epsilon^{loc}(V_S)$.

In Section 6 we will prove the following theorem.

Theorem 2.1. Suppose $D \in \mathcal{M}(\mathbb{R}^2)$ and $0 < \epsilon < \infty$. Then

$$D \in \mathbf{n}_\epsilon^{loc}(V_S) \;\Leftrightarrow\; \begin{cases} \Sigma(D, T_\epsilon) = 0 & \text{if } \Phi(\epsilon) < 0; \\ \Sigma(D, T_\epsilon) = 0 \text{ or } \Sigma(D, \emptyset) = 0 & \text{if } \Phi(\epsilon) = 0; \\ \Sigma(D, \emptyset) = 0 & \text{if } \Phi(\epsilon) > 0; \end{cases}$$

It will turn out that Theorem 4.2 will follow rather directly from this Theorem together with Theorems 1.6.1 and 1.6.2 of [AW1].

3 Some useful definitions and notations.

Whenever $a \in \mathbb{R}^2$ and $0 < r < \infty$ we let

$$\mathbf{C}(a,r) = \{x \in \mathbb{R}^2 : |x-a| = r\} = \mathbf{bdry}\,\mathbf{U}(a,r).$$

We let

$$\mathbf{int}, \quad \mathbf{cl}, \quad \text{and} \quad \mathbf{bdry}$$

stand for "interior", "closure" and "boundary", respectively.

We let

$$\mathbf{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$$

and we let

$$\mathbf{e}_1 = (1,0) \in \mathbf{S}^1, \quad \mathbf{e}_2 = (0,1) \in \mathbf{S}^1.$$

We let

$$\mathcal{H}^1$$

be one dimensional Hausdorff measure on $\mathbb{R}^2$.

Whenever $A \subset \mathbb{R}^2$ and a is an accumulation point of A we let

$$\mathbf{Tan}(A,a) = \bigcap_{0<r<\infty} \mathbf{cl}\,\{t(x-a) : 0 < t < \infty \text{ and } x \in A \cap (\mathbf{B}(a,r) \sim \{a\})\}$$

and we let

$$\mathbf{Nor}(A,a) = \bigcap_{w \in \mathbf{Tan}(A,a)} \{v \in \mathbb{R}^2 : v \bullet w \le 0\}.$$

Whenever $a \in \mathbb{R}^2$ and $v \in \mathbb{R}^2 \sim \{0\}$ we set

$$\mathbf{h}(a,v) = \{x \in \mathbb{R}^2 : (x-a) \bullet v \le 0\} \quad \text{and} \quad \mathbf{l}(a,v) = \{x \in \mathbb{R}^2 : (x-a) \bullet v = 0\}.$$

For each $\theta \in \mathbb{R}$ we let

$$\mathbf{u}(\theta) = (\cos\theta, \sin\theta).$$

Whenever $a, b \in \mathbb{R}^2$ we let

$$(a,b) = \{(1-t)a+tb : 0 < t < 1\} \quad \text{and we let} \quad [a,b] = \{(1-t)a+tb : 0 \le t \le 1\}.$$

4 The sets T_r, $0 < r \le \infty$.

Let T_r, $0 < r < \infty$, and R be as in Section 1.3

Suppose $r \in (0, R]$. Let

$$B_r = \mathbf{bdry}\, T_r \quad \text{and let} \quad \delta_r(x) = \mathbf{dist}\,(x, B_r) \quad \text{for } x \in \mathbb{R}^2.$$

Let

$$\begin{aligned} U_r &= \{x \in \mathbb{R}^2 : \delta_r(x) < r\}; \\ \xi_r &= \{(x,b) \in U_r \times B_r : |x-b| = \delta_r(x)\}; \\ \nu_r &= \{(b,u) \in B_r \times \mathbf{S}^1 : \mathbf{B}(b-ru, r) \subset T_r\}. \end{aligned}$$

Let

$$\mathcal{B}_r$$

be the family of connected components of $B_r \sim \mathbf{bdry}\, S$ and let

$$\gamma_r = \sup \left\{ \frac{\mathcal{H}^1(A)}{r} : A \in \mathcal{B}_r \right\}.$$

4.1 Basic theory for T_r, $0 < r < R$.

Proposition 4.1. Suppose $0 < r \leq R$. The following statements hold:

(i) C_r and T_r are compact convex subsets of S.

(ii) B_r is a continuously differentiable compact one dimensional submanifold of $\mathbb{R}^2$.

(iii) ν_r is a function with domain B_r.

(iv) $\mathbf{Nor}(T_r, b) = \{t\nu_r(b) : 0 \leq t < \infty\}$ and $\mathbf{Nor}(B_r, b) = \{t\nu_r(b) : t \in \mathbb{R}\}$ for any $b \in B_r$.

(v) For any $y, b \in B_r$ we have

$$|\nu_r(y) - \nu_r(y)| \leq \frac{|y-b|}{r}.$$

(vi) ξ_r is a function with domain U_r which retracts U_r onto B_r.

(vii) For any $x \in U_r$ we have

$$x = \begin{cases} \xi_r(x) + \delta_r(x)\nu_r(\xi_r(x)) & \text{if } x \in U_r \sim T_r, \\ \xi_r(x) - \delta_r(x)\nu_r(\xi_r(x)) & \text{if } x \in U_r \cap T_r. \end{cases}$$

(viii) Whenever $0 < s < r$, $x, a \in U_r$ and $\max\{\delta_r(x), \delta_r(a)\} \leq s$ we have

$$|\xi_r(x) - \xi_r(a)| \leq \frac{r}{r-s}|x-a|.$$

Remark 4.1. See [FE2]. In particular, (vi) implies that if $0 < r < \infty$ then the reach of B_r in the sense of [FE2] is at least r. The proof of (viii) was inspired by the proof of [FE2, 4.8(8)].

Proof. It is evident that C_r and T_r are compact since S is compact. If $c, c' \in C_r$ then the convex hull of $\mathbf{B}(c,r) \cup \mathbf{B}(c',r)$ equals $\cup\{\mathbf{B}((1-t)c+tc', r) : 0 \leq t \leq 1\}$ and is a subset of S; the convexity of C_r and T_r follow. Thus (i) holds.

Suppose $(b, u) \in B_r \times \mathbf{S}^1$. It is evident that

$$(b,u) \in \nu_r \;\Leftrightarrow\; (b-y) \bullet u \leq \frac{1}{2r}|y-b|^2 \quad \text{for } y \in B_r. \tag{3}$$

Suppose $(b,u) \in \nu_r$. Then

$$\mathbf{h}(b,u) = \mathbf{Tan}(\mathbf{B}(b - ru, r), b) \subset \mathbf{Tan}(T_r, b);$$

since T_r is convex we infer that

$$\mathbf{Tan}(T_r, b) = \mathbf{h}(b,u). \tag{4}$$

Thus (iv) holds and ν_r is a function.

Suppose $b \in B_r$. Choose sequences a in T_r and c in C_r such that $a_i \in \mathbf{B}(c_i, r)$ for each positive integer i and $a_i \to b$ as $i \to \infty$. Passing to a subsequence if necessary we obtain $c_\infty \in C_r$ such that $c_i \to c_\infty$ as $i \to \infty$. It follows that $|b - c_\infty| \le r$ and $\mathbf{B}(c_\infty, q) \subset T_r$ whenever $0 < q < r$ which in turn implies that $\mathbf{B}(c_\infty, r) \subset T_r$ since T_r is closed. Were it the case that $|b - c_\infty| < r$ the point b would have to be interior to T_r so $|b - c_\infty| = r$. It follows that $\mathbf{B}(b - ru, r) \subset T_r$ where $u \in \mathbf{S}^1$ is such that $b - c_\infty = |b - c_\infty| u$. Thus b is in the domain of ν_r.

Suppose $b, y \in B_r$, $u = \nu_r(b)$, $v = \nu_r(y)$ and $z = (b - ru) + rv \in T_r$. Then from (3), (4) and the convexity of T_r we infer that

$$\frac{r}{2}|v - u|^2 = r(v - u) \bullet v = (z - y) \bullet v + (y - b) \bullet v \le \frac{1}{2r}|y - b|^2$$

so (v) holds. It is an elementary consequence of the convexity of T_r that B_r is a rectifiable curve so, in view of (v), (ii) holds.

Suppose $a \in U_r \sim B_r$. Let $b \in B_r$ be such that $\delta_r(x) = |a - b|$. Then $\mathbf{Tan}(B_r, b) = \mathbf{l}(b, b - a) = \mathbf{l}(b, \nu_r(b))$. Since $\mathbf{B}(b - r\nu_r(b), r) \subset T_r$ and $\delta_r(a) < r$ we infer that $\mathbf{B}(a, \delta_r(a)) = \{b\}$. (vi) and (vii) follow.

Suppose s, x, a are as in (viii). Then, following the proof of , we find that

$$\begin{aligned}
&|x - a||\xi_r(x) - \xi_r(a)| \\
&\quad \ge (x - a) \bullet (\xi_r(x) - \xi_r(a)) \\
&\quad = [(x - \xi_r(x)) - (a - \xi_r(a)) + (\xi_r(x) - \xi_r(a))] \bullet (\xi_r(x) - \xi_r(a)) \\
&\quad \ge \left(-\frac{\rho_r(x)}{2r} - \frac{\rho_r(a)}{2r} + 1 \right) |\xi_r(x) - \xi_r(a)|^2
\end{aligned}$$

so (viii) holds. □

Theorem 4.1. The following statements hold:

(i) $\gamma_R \le \pi/2$.

(ii) $\gamma_r < \pi/2$ if $0 < r < R$.

(iii) Each member of $\mathcal{B}_r$ is an arc of a circle of radius r for each $r \in (0, R]$.

(iv) If $0 < q < r < R$ then

$$\sup\{\delta_r(x) : x \in B_q\} \le (\sec \gamma_r)(r - q).$$

(v) $(0, R] \ni r \mapsto \gamma_r$ is increasing.

Proof. Suppose $0 < r \le R$ and $b \in C \in \mathcal{B}_r$. Let $c = b - r\nu_r(b)$ and A is the connected component of b in $\mathbf{C}(c, r)$. Were it the case that $\mathbf{C}(c, r) \cap \mathbf{bdry}\, S = \emptyset$ there would exist $s \in (r, \infty)$ such that $\mathbf{U}(c, s) \subset S$ and this would force b to be interior to T_r. Applying a rigid motion to S if necessary we may assume $c = 0$ and that there are $\gamma \in (0, \pi)$ and $\alpha \in [0, \gamma)$ such that $b = r\mathbf{u}(\alpha)$, $\nu_r(b) = \mathbf{u}(\alpha)$, $\mathbf{B}(0, r) \subset S$, $d_\pm = r\mathbf{u}(\pm\gamma) \in C \cap \mathbf{bdry}\, S$, and

$$A = \{r\mathbf{u}(\theta) : |\theta| < \gamma\} \cap \mathbf{bdry}\, S = \emptyset.$$

Owing to the convexity of S we have

$$S \subset W = \mathbf{h}(d_-, \nu_r(d_-)) \cap \mathbf{h}(d_+, \nu_r(d_+)).$$

Suppose γ were greater that $\pi/2$. Choose $\beta \in (\pi/2, \gamma)$ and note that compact set $\{r\mathbf{u}(\theta) : |\theta| \le \beta\}$ does not meet $\mathbf{bdry}\, S$. Thus

$$\mathbf{B}(t\mathbf{e}_1, |r\mathbf{u}(\beta) - t\mathbf{e}_1|) \subset S$$

for t sufficiently small and positive and, since $|r\mathbf{u}(\beta) - t\mathbf{e}_1| > r$ whenever $0 < t < \infty$, this forces b to be interior to B_r. Thus $\gamma \le \pi/2$.

Suppose $\gamma = \pi/2$. Then $d_\pm = \pm r\mathbf{e}_2$ and $W = \{x \in \mathbb{R}^2 : |x_2| \le r\}$. Thus no ball of radius greater than r is contained in S so $r = R$. Let $b' \in \mathbf{cl}\, C$ be such that $|b'| \ge |x|$ whenever $x \in C$ and let $s = |b'| > R$. Then $b' = s\nu_r(b)$ and $\mathbf{B}((s - R)\nu_r(b), R) \subset S \subset W$. Thus $\nu_r(b) = \mathbf{e}_1$. But this implies b is interior to T_r. Thus, in case $\gamma = \pi/2$, $C = A$.

Suppose $0 < q \le r < R$, let $C' = W \cap (B_q \sim \mathbf{bdry}\, S)$ and let $b' \in C'$ be such that

$$|b'| \ge |x| \quad \text{whenever } x \in C'.$$

Note that $|b'| \ge r$ since otherwise b' would be interior to $\mathbf{B}(0, r)$ which is a subset of T_q. Let $\beta \in (-\gamma, \gamma)$ be such that $b' = |b'|\mathbf{u}(\beta)$ and note that $\nu_q(b') = \mathbf{u}(\beta)$. Since $\mathbf{B}(b' - q\mathbf{u}(\beta), q) \subset S \subset W$ we find that

$$(((b' - q\mathbf{u}(\beta)) + q\mathbf{u}(\pm\gamma)) - r\mathbf{u}(\pm\gamma)) \bullet \mathbf{u}(\pm\gamma) \le 0$$

which amounts to

$$(|b'| - q)\cos(\pm\gamma - \beta) \le r - q. \tag{5}$$

In case $\beta \ge 0$ we have $\cos\gamma \le \cos(\gamma - \beta)$ and in case $\beta < 0$ we have $\cos\gamma \le \cos(-\gamma - \beta)$ so that (5) implies

$$(|b'| - q)\cos\gamma \le r - q.$$

Suppose $q = r$. Were it the case that $|b'| > r$ we would have $\beta \ne \gamma$ which is impossible in view of (5). Thus $C = A$. Moreover γ equals the length of C divided by r. Thus we have established (i)-(iv).

Suppose $0 < q < r \le R$ and $C' \in \mathcal{B}_q$ and $C' \subset W$. Then there are c' and $d'_\pm \in \mathbf{bdry}\, S \cap \mathbf{cl}\, C'$ such that $d'_\pm - c' = q\nu_q(d'_\pm)$. The convexity of S implies that the length of C' divided by q is less than the length of C divided by r and this establishes (v). □

Proposition 4.2. We have

$$\lim_{r \uparrow R} \sup\{\delta_R(x) : x \in B_r\} = 0.$$

Proof. Suppose the Proposition were false. Then there would exist a sequences r in $(0, R)$ and b, d in S such that $b_i \in B_{r_i}$, $d_i \in B_R$ and $|b_i - d_i| = \sup\{\delta_R(x) : x \in B_{r_i}\}$ for each positive integer i, such that $\lim_{i\to\infty} r_i = R$ but such that $\eta = \lim_{i\to\infty} |b_i - d_i| > 0$.

Suppose i is a positive integer. We have $b_i - d_i \in \mathbf{Nor}(T_{r_i}, b_i)$ so $b_i - d_i = |b_i - d_i|\nu_{r_i}(b_i)$. We we also have $b_i - d_i \in \mathbf{Nor}(B_R, d_i)$ since $\mathbf{Tan}(B_R, d_i)$ is a line so $\nu_R(d_i) = \nu_{r_i}(b_i)$. Thus

$$\mathbf{B}(d_i - R\nu_R(d_i), R) \cup \mathbf{B}(b_i - r_i\nu_R(d_i), r_i) \subset S.$$

Passing to a subsequence if necessary we obtain $b_\infty \in S$, $d_\infty \in B_R$ and $u_\infty \in \mathbf{S}^1$ such that $(b_i, d_i, \nu_R(d_i)) \to (b_\infty, d_\infty, u_\infty)$ as $i \to \infty$ which implies that $b_\infty = d_\infty + \eta u_\infty$. Since S is closed we have that that $Z_\infty \subset S$ where Z_∞ is the convex hull of $\mathbf{B}(d_\infty - Ru_\infty, R) \cup \mathbf{B}(b_\infty - Re_\infty u_\infty, R)$. Since $d_\infty \in \mathbf{int}\, Z_\infty \subset S$ we infer that $d_\infty \in \mathbf{int}\, T_R$ which is incompatible with $d_\infty \in B_R$. □

Corollary 4.1. For any $r \in (0, R]$ we have

$$\lim_{(0,R] \in s \to r} \sup\{\delta_r(x) : x \in B_s\} = 0.$$

Proof. When $r = R$ this is the statement of the preceding Proposition. If $0 < r < R$ this is a straightforward consequence of Theorem 4.1. □

4.2 Proof of Proposition 1.1.

The continuity of Φ follows from Proposition 4.1 and Corollary 4.1.

Suppose $0 < r < R$. By Corollary 4.1 there is $\eta > 0$ such that

$$r - \eta < s < r + \eta \ \Rightarrow 0 < s \leq R \text{ and } T_r \sim T_s \subset U_r.$$

Suppose $r < s < r + \eta$. For each $A \in \mathcal{B}_s$ we let c_A be the center of the circle containing A, we let

$$B_{r,A} = (B_r \sim T_s) \cap \{c_A + t(x - c_A) : 1 < t < \infty \text{ and } x \in A\}$$

and we let

$$T_{r,A} = (T_r \sim T_s) \cap \{c_A + t(x - c_A) : 1 < t < \infty \text{ and } x \in A\}.$$

Then

$$B_r \sim T_s = \cup_{A \in \mathcal{B}_s} B_{r,A} \quad \text{and} \quad T_r \sim T_s = \cup_{A \in \mathcal{B}_s} T_{r,A}.$$

Let $X_r : U_r \to \mathbb{R}^2$ be such that

$$X_r(x) = \nu_r(\xi_r(x)) \quad \text{for } x \in U_r.$$

Then X is Lipschitzian and

$$0 < \mathbf{div}\, X(x) \le \frac{1}{r - \Delta_s}$$

whenever $x \in T_r \sim T_s$ and X is differentiable at x and where we have set $\Delta_s = \sup\{\delta_r(x) : x \in B_s\}$.

Suppose $A \in \mathcal{B}_s$. Then

$$\mathcal{H}^1(B_{r,A}) = \int_{B_{r,A}} X \bullet \nu_r \, d\mathcal{H}^1 \quad \text{and} \quad \mathcal{H}^1(A) \ge \int_A X \bullet \nu_s \, d\mathcal{H}^1$$

and follows from the Gauss-Green Theorem (see [FE1, 4.5.6]) that

$$\begin{aligned} \mathcal{H}^1(A) - \mathcal{H}^1(B_{r,A}) &\ge \left(\int_A X \bullet \nu_s \, d\mathcal{H}^1 - \int_{B_{r,A}} X \bullet \nu_r \, d\mathcal{H}^1 \right) \\ &= - \int_{T_{r,A}} \mathbf{div}\, X \, d\mathcal{L}^2 \\ &\ge - \frac{1}{r - \Delta_s} \mathcal{L}^2(T_{r,A}). \end{aligned}$$

Summing over $\mathcal{B}_s$ we find that

$$\mathbf{TV}(T_s) - \mathbf{TV}(T_r) \ge - \left(\frac{1}{r - \Delta_s} \right) \mathcal{L}^2(T_r \sim T_s).$$

Consequently,

$$\begin{aligned} \Phi(s) - \Phi(r) &= \mathbf{TV}(T_s) - \mathbf{TV}(T_r) + \frac{\mathcal{L}^2(T_r \sim T_s)}{r} + \left(\frac{1}{r} - \frac{1}{s} \right) \mathcal{L}^2(T_s) \\ &\ge \left(\frac{1}{r} - \frac{1}{r - \Delta_s} \right) \mathcal{L}^2(T_r \sim T_s) + \frac{s - r}{rs} \mathcal{L}^2(T_s) \\ &= - \frac{\Delta_s}{r(r - \Delta_s)} \mathcal{L}^2(T_r \sim T_s) + \frac{s - r}{rs} \mathcal{L}^2(T_s). \end{aligned}$$

It follows from Theorem 4.1 that

$$\limsup_{s \downarrow r} \frac{\mathcal{L}^2(T_r \sim T_s)}{s - r} < \infty$$

and it follows from Proposition 4.1 that $\lim_{s \downarrow r} \Delta_s = 0$. Thus

$$\liminf_{s \downarrow r} \frac{\Phi(s) - \Phi(r)}{s - r} \ge \frac{\mathcal{L}^2(T_r)}{r^2}.$$

5 Some facts about $\mathcal{C}_1(\mathbb{R}^2)$.

We will need some results on the space $\mathcal{C}_1(\mathbb{R}^2)$ which was defined in [AW1, 1.5.1].

We let

$$\mathbf{w}(m) = \sqrt{1+m^2} \quad \text{for } m \in \mathbb{R}.$$

Note that if I is an open interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable then $(\mathbf{w}' \circ f')'(t)$ is the curvature of f at $(t, f(t))$ whenever $t \in I$.

5.1 A "thickness" theorem.

Lemma 5.1. Suppose

(i) $E \in \mathcal{C}_1(\mathbb{R}^2)$ and E equals the support of the generalized function corresponding to 1_E;

(ii) $a \in \mathbf{bdry}\, E$, $u, v \in \mathbf{S}^1$, $u \in \mathbf{Tan}(\mathbf{bdry}\, E, a)$ and $-v \in \mathbf{Nor}(E, a)$;

(iii) C is the connected component of a in $\{x \in \mathbf{bdry}\, E : |(x-a) \bullet u| < 1\}$.

Then there is a continuously differentiable function

$$f : (-1, 1) \to \mathbb{R}$$

such that

(iv) $|f(t)| \le 1 - \sqrt{1-t^2}$ and $|f'(t)| \le |t|/\sqrt{1-t^2}$ whenever $-1 < t < 1$;

(v) $\mathbf{Lip}\,(\mathbf{w}' \circ f') \le 1$;

(vi) $C = \{a + tu + f(t)v : -1 < t < 1\}$.

Proof. We may assume without loss of generality that $a = 0$, $u = \mathbf{e}_1$ and $v = \mathbf{e}_2$.

Let $\mathcal{G}$ be the family of ordered pairs (J, g) such that J is a subinterval of $(-1, 1)$ containing 0, $g : J \to \mathbb{R}$ is continuously differentiable, and $g \subset C$. Owing to the regularity of $\mathbf{bdry}\, E$ as stated in [AW1, 1.5.1] we find that if $(J_i, g_i) \in \mathcal{G}$, $i = 1, 2$, then $(J_1 \cup J_2, g_1 \cup g_2) \in \mathcal{G}$ from which it follows that there is $(I, f) \in \mathcal{G}$ such that $g \subset f$ whenever $(J, g) \in \mathcal{G}$.

By [AW1, 5.4.1] we find that $\mathbf{Lip}\,(\mathbf{w}' \circ f') \le 1$. This implies that

$$\mathbf{w}'(|f'(t)|) = |\mathbf{w}'(f'(t)) - \mathbf{w}'(f'(0))| \le |t| \quad \text{for } t \in I;$$

since $\mathbf{w}'$ is increasing we find that

$$|f'(t)| \le \mathbf{v}(|t|) = \frac{|t|}{\sqrt{1-t^2}} \quad \text{for } t \in I \tag{6}$$

where $\mathbf{v}$ is the function inverse to $\mathbf{w}'$. This in turn implies that

$$|f(t)| \le 1 - \sqrt{1-t^2} \quad \text{for } t \in I. \tag{7}$$

Let $t_L = \inf I$ and let $t_R = \sup I$. Owing to (6) we find that the limits

$$x_L = \lim_{t \downarrow t_L} (t, f(t)) \quad \text{and} \quad x_R = \lim_{t \uparrow t_R} (t, f(t))$$

exist and are in $\mathbf{bdry}\, E$. Owing to the regularity properties of $\mathbf{bdry}\, E$ and the estimate (6) we find that if either $-1 < t_L$ or $t_R < 1$ the maximality of I is contradicted. □

Theorem 5.1. Suppose $C \in \mathcal{C}_1(\mathbb{R}^2)$, C is compact and convex, $a \in \mathbf{bdry}\, C$ and

$$v \in \mathbf{S}^1 \cap \mathbf{Nor}(C, a).$$

Then

$$\mathbf{B}(a - v, 1) \subset C.$$

Proof. It will suffice to consider the case $a = 0$ and $v = -\mathbf{e}_2$.

Let $a^- = \inf\{x_1 : x \in C\}$, $a^+ = \sup\{x_1 : x \in C\}$ and $b = \sup\{x_2 : x \in C\}$ and let

$$f^\pm : [a^-, a^+] \to [0, b]$$

be such that

$$f^-(t) = \inf\{u : (t, u) \in C\} \quad \text{and} \quad f^+(t) = \sup\{u : (t, u) \in C\}$$

for $a^- \leq t \leq a^+$. Then f^- is convex, f^+ is concave, $f^- \leq f^+$ and

$$C = \{(x_1, x_2) \in [a^-, a^+] \times \mathbb{R} : f^-(x_1) \leq x_2 \leq f^+(x_1)\}.$$

Let c be such that $(c, b) \in C$.

Applying Lemma 5.1 with $a = 0$, $u = \mathbf{e}_1$ and $v = \mathbf{e}_2$ we find that $a^- \leq -1$, that $1 \leq a^+$, that

$$|(t, f^-(t)) - (0, 1)| \geq 1 \quad \text{for } a^- \leq t \leq a^+.$$

Applying Lemma 5.1 with $a = (a^+, f^-(a^+))$, $u = \mathbf{e}_2$ and $v = -\mathbf{e}_1$ we find that $f^-(a^+) \geq 1$, that $b \geq f^+(a^+) + 1 \geq f^-(a^+) + 1 \geq 2$, that

$$|(t, f^-(t)) - (a^+ - 1, f^-(a^+))| \geq 1 \quad \text{for } a^+ - 1 \leq t \leq a^+.$$

and that

$$|(t, f^+(t)) - (a^+ - 1, f^+(a^+))| \geq 1 \quad \text{for } a^+ - 1 \leq t \leq a^+.$$

Applying Lemma 5.1 with $a = (a^-, f^-(a^-))$, $u = \mathbf{e}_2$ and $v = \mathbf{e}_1$ we find that $f^-(a^-) \geq 1$, that $b \geq f^+(a^-) + 1 \geq f^-(a^-) + 1 \geq 2$, that

$$|(t, f^-(t)) - (a^- + 1, f^-(a^-))| \geq 1 \quad \text{for } a^- \leq t \leq a^- - 1.$$

and that

$$|(t, f^+(t)) - (a^- + 1, f^+(a^-))| \geq 1 \quad \text{for } a^- \leq t \leq a^- - 1.$$

That the Theorem holds should now be clear. □

6 Proof of Theorem 2.1.

In this section we prove Theorem 2.1. Owing to the way various quantities change with respect to transformation by homotheties, *we may assume that* $\epsilon = 1$.

Suppose $D \in \mathcal{M}(\mathbb{R}^2)$, $\mathcal{L}^2(D) > 0$ and

$$D \in \mathbf{n}_1^{loc}(V_S).$$

We need to show that

$$\Sigma(D, T_1) = 0.$$

Owing to the regularity results of [AW1, 1.5.1] *we may assume that* D *equals the support of the generalized function corresponding to* 1_D. From [AW1, Theorem 10.1] we find that

$$D \subset S.$$

We also know from [AW1, 1.5.1] that for each $b \in \mathbf{bdry}\, D$ there are open intervals I and J containing 0; a continuously differentiable function $g : I \to J$; and an isometry $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$g(0) = 0, \quad g'(0) = 0, \quad \Phi(b) = 0 \tag{8}$$

$$\Phi[E] \cap (I \times J) = \{(u, v) \in I \times J : v \leq g(u)\}. \tag{9}$$

Let $\mathcal{A}$ be the family of connected components $\mathbf{bdry}\, D \cap \mathbf{int}\, S$. We know from [AW1, 8.1] that if $A \in \mathcal{A}$ then A is an open arc of a circle of radius 1 and that the length of A does not exceed π; moreover, if $a \in A$ and c is the center of the circle containing A there is $\delta > 0$ such that

$$\mathbf{B}(c, 1) \cap \mathbf{U}(a, \delta) = D \cap \mathbf{U}(a, \delta). \tag{10}$$

For each $A \in \mathcal{A}$ let

$$\mathbf{ends}(A) = (\mathbf{cl}\, A) \sim A$$

and note that $\mathbf{ends}(A)$ has exactly two members.

6.1 The proof.

The Theorem will follow from the following Lemmas and Propositions.

Proposition 6.1. Suppose $a \in D \cap (\mathbf{bdry}\, S)$. Then $a \in \mathbf{bdry}\, D$ and

$$D \subset S \subset a + \mathbf{Tan}(D, a).$$

Moreover, if $A \in \mathcal{A}$, $a \in \mathbf{ends}(A)$ and c is the center of the circle containing A then

$$\mathbf{Tan}(D, a) = \mathbf{h}(a, a - c).$$

Proof. Since $D \subset S$ we infer that $\mathbf{Tan}(D,a) \subset \mathbf{Tan}(S,a)$. Since $a \in \mathbf{bdry}\, S$ and S is convex $\mathbf{Tan}(S,b)$ is contained in a closed halfspace. Thus $\mathbf{Tan}(S,b) = \mathbf{Tan}(D,b)$. This in turn implies $b \in \mathbf{bdry}\, D$. The final assertion of the Proposition follows directly from (8) and (9). □

Lemma 6.1. Suppose $A_i \in \mathcal{A}$, $a_i \in \mathbf{ends}(A_i)$ and c_i is the center of the circle containing A_i, $i = 1,2$. Then

$$(a_i - c_i) \bullet (c_j - c_i) \le 0 \quad \text{whenever } \{i,j\} = \{1,2\}.$$

Proof. Suppose $\{i,j\} = \{1,2\}$. From the preceding Proposition and the fact that $|a_i - c_i| = |a_j - c_j|$ we obtain

$$\begin{aligned} 0 &\ge (a_j - a_i) \bullet (a_i - c_i) \\ &= (c_j + (a_j - c_j) - (c_i + (a_i - c_i)) \bullet (a_i - c_i) \\ &= (c_j - c_i) \bullet (a_i - c_i) + (a_j - c_j) \bullet (a_i - c_i) - |a_i - c_i|^2 \\ &\ge (c_j - c_i) \bullet (a_i - c_i). \end{aligned}$$

□

Proposition 6.2. Suppose a, b, c, A are such that $A \in \mathcal{A}$, $A \subset \mathbf{C}(c,r)$, $\mathbf{ends}(A) = \{a,b\}$; and $V = \{t(x-c) : x \in \mathbf{cl}\, A \text{ and } t > 1\}$.

Then exactly one of the following holds:

(i) $D \cap V = \emptyset$.

(ii) The length of A equals π and there are a', b', c', A' such that $A' \in \mathcal{A}$, $A' \subset \mathbf{C}(c', r)$, $\mathbf{ends}(A') = \{a', b'\}$ and such that, for some $q > 2$,

$$a' = a + q(d-c), \quad b' = b + q(d-c), \quad c' = c + q(d-c)$$

where d is the midpoint of A.

Furthermore, if (ii) holds we have

$$D \cap G = \emptyset \quad \text{and} \quad G \subset S$$

where G is the union of the segments (e, e') such that $e \in A$, $e' \in A'$ and (e, e') is parallel to the line containing c and c'.

Proof. Since S is compact and convex, whenever L is a line and $y \in L \cap \mathbf{int}\, S$ there are unique x, z, t such that $\{x, z\} \subset L \cap \mathbf{bdry}\, S$, $0 < t < 1$ and $y = (1-t)x + tz$. Since $A \subset \mathbf{int}\, S$ it follows that there is one and only one function $\upsilon : A \to \mathbf{int}\, V \cap \mathbf{bdry}\, S$ such that

$$\mathbf{bdry}\, S \cap \{c + t(x-c) : 1 < t < \infty\} = \{\upsilon(x)\} \quad \text{for } x \in A.$$

Owing to the convexity of S we find that

(iii) $\qquad c + u \in A \quad$ whenever $x \in A$ and and $u \in \mathbf{S}^1 \cap \mathbf{Nor}(S, \upsilon(x))$.

Suppose $D \cap V \neq \emptyset$. Owing to the compactness and regularity properties of D there exists $d' \in V \cap \mathbf{bdry}\, D$ such that

$$|d' - c| = \min\{|x - c| : x \in D \cap V\} > 1.$$

It follows from (10) that

$$\text{(v)} \qquad \mathbf{Nor}(D, d') = \{t(c - d') : 0 \leq t < \infty\}.$$

From Proposition 6.1 we have

$$d' \in D \subset \mathbf{h}(a, a - c) \cap \mathbf{h}(b, b - c).$$

Since $d' \notin \{a, b\}$ this implies $d' \in \mathbf{int}\, V$ so there is $d \in A$ such that $d' = c + t(d - c)$ for some $t \in (1, \infty)$.

Were it the case that $d' \in \mathbf{bdry}\, S$ we could infer from the Proposition 6.1 that $\mathbf{Nor}(S, d') = \mathbf{Nor}(D, d')$ which is incompatible with (iii). Thus $d' \notin \mathbf{bdry}\, S$. Since $d' \in D \subset S$ we infer that $d' \in \mathbf{int}\, S$. Thus there is $A' \in \mathcal{A}$ such that $d' \in A'$. It follows from (10) that $A' \subset \mathbf{C}(c', 1)$ where $c' = c + u(d - c)$ for some $u \in (t + r, \infty)$. Thus $|c' - c| > 2$.

Let a', b' such that $\mathbf{ends}(A') = \{a', b'\}$. From the preceding Proposition we have

$$(a' - c') \bullet (c - c') \leq 0 \quad \text{and} \quad (b' - c') \bullet (c - c') \leq 0.$$

Since the length of A' does not exceed π we infer that A' is a semicircle with midpoint d'. By a similar argument we find that A is a semicircle with midpoint d. (ii) now follows.

Let G be as in the final conclusion of the Proposition. From the convexity of S we infer that the rectangle containing the points a, b, a', b' is a subset of S so $G \subset \mathbf{int}\, S$. Suppose, contrary to the last conclusion of the Proposition, $p \in D \cap G$. Since $p \notin \mathbf{bdry}\, S$ there is $B \in \mathcal{A}$ such that $p \in B$. Since B cannot meet $A \cup A'$ we must have $\mathbf{ends}(B) \subset [a, a'] \cup [b, b']$. Since $(d, d') \cap \mathbf{bdry}\, D = \emptyset$ we infer that *either* $\mathbf{ends}(B) \subset [a, b]$ *or* $\mathbf{ends}(B) \subset [a', b']$. This is impossible since $B \in \mathcal{A}$ and B meets either (a, b) or (a', b') tangentially. □

Lemma 6.2. Suppose a, b, c, A are such that $A \in \mathcal{A}$, $A \subset \mathbf{C}(c, 1)$, $\mathbf{ends}(A) = \{a, b\}$; and $V = \{t(x - c) : x \in \mathbf{cl}\, A \text{ and } t > 1\}$.

Then $D \cap V = \emptyset$.

Proof. Suppose $D \cap V \neq \emptyset$. Then there are A', a', b', c' and G as in (ii) of the preceding Proposition.

Then $D \cap G = \emptyset$ and $D \cup G \subset S$ so

$$V_S(D \cup G) - V_S(D) = -\mathcal{L}^2(G) = \pi - 2|c - c'|;$$

moreover,

$$\mathbf{TV}(D \cup G) - \mathbf{TV}(D) = -2\pi + 2|c - c'|.$$

It follows that

$$0 \leq (\mathbf{TV}(D \cup G) + V_S(D \cup G) - (\mathbf{TV}(D) + V_S(D)) = -\pi < 0$$

which is a contradiction. □

Lemma 6.3. D is convex.

Proof. Suppose $e \in \mathbf{bdry}\, D$.

If $e \in \mathbf{bdry}\, S$ then $D \subset S \subset e + \mathbf{Tan}(S, e)$.

Suppose $e \notin \mathbf{bdry}\, S$. Then $e \in \mathbf{int}\, S$ so there is $A \in \mathcal{A}$ such that $e \in A$. Let a, b, c be such that $A \subset \mathbf{C}(c, 1)$ and $\mathbf{ends}(A) = \{a, b\}$. Let $V = \{t(x - c) : x \in \mathbf{cl}\, A$ and $t > 1\}$. We have $D \cap V = \emptyset$ from the preceding Lemma. Since $D \subset S \subset \mathbf{h}(a, a - c) \cap \mathbf{h}(b, b - c)$ by Proposition 6.1 we find that $D \subset e + \mathbf{Tan}(\mathbf{B}(c, 1), e)$.

It follows that D is convex. □

Lemma 6.4. $D \subset T_1$.

Proof. Suppose $d \in D$. Let $a \in \mathbf{bdry}\, D$ be such that $|d - a| = \mathbf{dist}\,(d, \mathbf{bdry}\, D)$. In case $|d - a| \geq 1$ we have $\mathbf{B}(d, 1) \subset \mathbf{B}(d, |d - a|) \subset D \subset S$ so $d \in T_1$.

Suppose $|d - a| < 1$. Letting $v \in \mathbf{S}^1$ be such that $d - a = -|d - a|v$ we find that $v \in \mathbf{Nor}(D, a)$ we have from Theorem 5.1 that $\mathbf{B}(a - v, 1) \subset D \subset S$. Since $|(a - v) - d| = 1 - |d - a| < 1$ we have $d \in T_1$. □

Lemma 6.5. Suppose $u \in \mathbf{S}^1$. Then

$$\sup\{x \bullet u : x \in S\} - \inf\{x \bullet u : x \in S\} > 2.$$

Proof. Let $b^- = \inf\{x \bullet u : x \in D\}$ and let $b^+ = \sup\{x \bullet u : x \in D\}$. Suppose the Lemma were false. Then, as $D \subset S$, $b^+ - b^- \leq 2$. Let $v \in \mathbf{S}^1$ be such that $u \bullet v = 0$. Let $a^- = \inf\{x \bullet v : x \in D\}$ and let $a^+ = \sup\{x \bullet v\} : x \in D\}$. Then $\mathcal{L}^2(D) \leq (a^+ - a^-)(b^+ - b^-) \leq 2(a^+ - a^-)$. Since D is convex by the preceding Lemma we have that $\mathbf{TV}(D) > 2(a^+ - a^-)$. Thus

$$\mathbf{TV}(D) + V_S(D) = \mathbf{TV}(D) - \mathcal{L}^2(D) > 0 = \mathbf{TV}(\emptyset) + V_S(\emptyset)$$

which is a contradiction. □

Lemma 6.6. $T_1 \subset D$.

Proof. Suppose, to the contrary, there were $e \in T_1 \sim D$. Let $d \in \mathbf{bdry}\, D$ be such that $\mathbf{U}(e, |d - e|) \cap D = \emptyset$. It follows that

$$d + \mathbf{Tan}(D, d) = \mathbf{h}(d, e - d). \tag{11}$$

Were it the case that $d \in \mathbf{bdry}\, S$ we would have $d - e \in \mathbf{Nor}(S, d)$ and so, by the convexity of S, $S \subset d + \mathbf{Tan}(S, d) = \mathbf{h}(d, d - e)\}$. But by $\mathbf{Tan}(S, d) = \mathbf{Tan}(D, d)$ by Proposition 6.1. Thus $d \in \mathbf{int}\, S$.

Let $A \in \mathcal{A}$ be such that $d \in A$ and let a, b, c be such that $A \subset \mathbf{C}(c, 1)$ and $\mathbf{ends}(A) = \{a, b\}$. Note that e and c are on opp Let $J = \mathbf{h}(a, a - c) \cap \mathbf{h}(b, b - c)$ and note that $S \subset J$ by Proposition 6.1. Since e belongs to a closed ball of radius 1 which is a subset of S we infer that the length of A equals π. Thus the lines $a + \mathbf{Tan}(\mathbf{bdry}\, D, a)$ and $b + \mathbf{Tan}(\mathbf{bdry}\, D, b)$ are parallel with distance 2 between them; this is excluded by Lemma 6.5. □

7 Proof of Theorem 1.1.

We now show that Theorem 2.1 and Theorems 1.6.1 and 1.6.2 of [AW1] imply Theorem 1.1.

For each $y \in (0,\infty)$ and $E \in \mathcal{M}(\mathbb{R}^2)$ we let

$$U_y(E) = \int_E \beta(y - 1_S(x))\, d\mathcal{L}^2 x = \beta(y-1)\mathcal{L}^2(E \cap S) + \beta(y)\mathcal{L}^2(E \sim S).$$

7.1 The proof.

The Theorem will follow from the following Lemmas and Propositions.

Lemma 7.1. Suppose $0 < \zeta < \infty$ and *either* $M = V_S$ *or* $0 < y < 1$ and $M = U_y$. Then

$$M(E \cap S) \le M(E) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).$$

Moreover,

$$D \in \mathbf{m}_\zeta(M) \;\Rightarrow\; \mathcal{L}^2(D \sim S) = 0.$$

Proof. The first assertion follows easily from [AW1, Proposition 2.2]and the second follows from [AW1, Proposition 10.2]. □

Lemma 7.2. Suppose $0 < y < 1$. Then

$$\mathbf{m}_\epsilon(U_y) = \mathbf{m}_{\eta(y)}(V_S).$$

Proof. Note that whenever $E \in \mathcal{M}(\mathbb{R}^2)$ we have

$$-\beta(y-1)\,(V_S)_{\eta(y)}\,(E) = (U_y)_\epsilon(E) \quad \text{if } \mathcal{L}^2(E \sim S) = 0. \tag{12}$$

Suppose $D \in \mathbf{m}_\epsilon(U_y)$. By the preceding Lemma with $M = U_y$ we find that $\mathcal{L}^2(D \sim S) = 0$ so for any $E \in \mathcal{M}(\mathbb{R}^2)$ we have

$$\begin{aligned}
-\beta(1-y)\,(V_S)_{\eta(y)}\,(D) &= (U_y)_\epsilon(D) \\
&\le (U_y)_\epsilon(E \cap S) \\
&= -\beta(1-y)\,(V_S)_{\eta(y)}\,(E \cap S) \\
&\le -\beta(1-y)\,(V_S)_{\eta(y)}\,(E)
\end{aligned}$$

where we have applied the preceding Lemma with $M = V_S$ to obtain the last inequality. Thus $D \in \mathbf{m}_{\eta(y)}\,(V_S)$.

By a similar argument one shows that $\mathbf{m}_{\eta(y)}\,(V_S) \subset \mathbf{m}_\epsilon(U_y)$. □

Lemma 7.3. Suppose $1 \le y < \infty$ and $0 < \zeta < \infty$. Then

$$D \in \mathbf{n}_\zeta^{loc}(U_y) \;\Rightarrow\; \mathcal{L}^2(D) = 0.$$

Proof. Suppose $D \in \mathbf{n}_\zeta^{loc}(U_y)$ and, contrary to the Lemma, $\mathcal{L}^2(D) > 0$. Then

$$\begin{aligned}(U_y)_\epsilon(D) &= \epsilon\mathbf{TV}(D) + \beta(y-1)\mathcal{L}^2(D \cap S) + \beta(y)\mathcal{L}^2(D \sim S)\\ &\geq \min\{\beta(y-1), \beta(y)\}\mathcal{L}^2(D)\\ &> 0\\ &= (U_y)_\epsilon(\emptyset).\end{aligned}$$

which is a contradiction. □

Suppose $f \in \mathbf{m}_\epsilon^{loc}(F)$. By [AW1, Theorem 1.6.1] we have

$$\{f > y\} \in \mathbf{n}_\epsilon^{loc}(U_y) \quad \text{whenever } 0 < y < \infty.$$

Let $Y = ||f||_{\mathbf{L}_\infty(\mathbb{R}^2)}$ and note that $0 \leq Y \leq 1$ by the preceding Lemma. If $0 < y < Y$ then $\mathcal{L}^2(\{f > y\}) > 0$ so $\Sigma(\{f > y\}, W_y) = 0$ and $\Psi(y) \leq 0$ by Theorem 2.1. If $Y \leq y < 1$ then $\mathcal{L}^2(\{f > y\}) = 0$ so $\Psi(y) = 0$. It follows that $Y \in J$ and (2) holds.

On the other hand, suppose f is as in (2). Then

$$\Sigma(\{f > y\}, W_y) = 0 \quad \text{if } 0 < y < Y$$

and

$$\mathcal{L}^2(\{f > y\}) = 0 \quad \text{if } Y \leq y < \infty.$$

From the preceding Lemma and Theorem 2.1 we infer that $\{f > y\} \in \mathbf{n}_\epsilon^{loc}(U_y)$ whenever $0 < y < \infty$. It follows from [AW1, Theorem 1.6.2] that $f \in \mathbf{m}_\epsilon^{loc}(F)$.

References

[AW1] W. K. Allard: *Total variation regularization for image denoising; I. Geometric theory.* to appear in SIAM Journal on Mathematical Analysis.

[AW2] W. K. Allard: *Total variation regularization for image denoising; III. Examples.* in preparation.

[CE] T. F. Chan and S.Esedoglu: *Aspects of Total variation Regularized* $\mathbf{L}^1$ *Function Approximation*, SIAM J. Appl. Math. 65:5 (2005), pp. 1817-1837.

[FE1] H. Federer: *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften in Einszeldarstellungen, Band 153, Springer Verlag, (1969)

[FE2] H. Federer: *Curvature measures*, Trans. Amer. Math. Soc., 95:3 (1959), pp. 418-491.

[ROF] L.Rudin, S.Osher, E.Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D. 60 (1992) 259-268.