Mathematics 103.07 **Test One** **October 5, 2006**

I have neither given nor received aid in the completion of this test.
Signature:

To get full credit you must show enough work to convince me that you know what you are doing!

The average on this test was 90.81 and the standard deviation was 23,27.

Problem	Possible	Grade
1	15	
2	15	
3	20	
4	15	
5	10	
6	10	
7	10	
8	20	
9	30	
10	45	
Total	190	

1. (a) (10 pts.) Find an equation for the plane in $\mathbf{R}^3$ containing the three points $(1,0,1),(0,1,1),(1,1,0)$.

(b) (5 pts.) Calculate the area of the triangle in $\mathbf{R}^3$ with vertices $(1,0,1),(0,1,1),(1,1,0)$.

Solution. Let $\mathbf{x}_0 = (1,0,1)$, $\mathbf{x}_1 = (0,1,1)$ and $\mathbf{x}_2 = (1,1,0)$. Let $\mathbf{a} = \mathbf{x}_1 - \mathbf{x}_0 = (-1,1,0)$ and let $\mathbf{b} = \mathbf{x}_2 - \mathbf{x}_0 = (0,1,-1)$. We have

$$\mathbf{a} \times \mathbf{b} = (-\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - \mathbf{k}) = -\mathbf{k} - \mathbf{j} - \mathbf{i} = -(1,1,1).$$

An equation for the plane is thus

$$(\mathbf{x} - \mathbf{x}_0) \bullet (\mathbf{a} \times \mathbf{b}) = 0 \quad \text{or} \quad x + y + z = 2$$

(Could've guessed it, right?) and the area of the triangle is

$$\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{\sqrt{3}}{2}.$$

2. Let P_1 be the plane which is the solution set of the equation $x + 2y + 3z = 4$ and let P_2 be the plane which is the solution set of the equation $2x + y + z = -2$.

(a) (10 pts). Exhibit a parametric representation for the line which is the intersection of P_1 and P_2.

(b) (5 pts.) Calculate the cosine of the angle between P_1 and P_2.

Solution. $\mathbf{n}_1 = (1, 2, 3)$ and $\mathbf{n}_2 = (2, 1, 1)$ are normals to P_1 and P_2 respectively. So a velocity vector $\mathbf{v}$ for the line of intersection (assuming it's not empty) is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (-1, 5, -3).$$

Let's look for a point (x_0, y_0, z_0) in $P_1 \cap P_2$ with $z_0 = 0$. This gives $x_0 + 2y_0 = 4$ and $2x_0 + y_0 = -2$ the unique solution of which is $x_0 = -8/3, y_0 = 10/3$. Thus

$$\mathbf{x}(t) = \frac{1}{3}(-8, 10, 0) + t(-1, 5, -3), \quad t \in \mathbf{R},$$

is a parametric representation for the line. The cosine of the angle between P_1 and P_2 is

$$\frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{7}{\sqrt{14}\sqrt{6}}.$$

3. (a) (10 pts.) Does

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4 + y^4}$$

exist? (Hint: Consider what happens on the lines $y = mx$.)
Solution. Let $f(x, y) = \frac{xy^3}{x^4+y^4}$ for $(x, y) \neq (0, 0)$. We have

$$f(x, mx) = \frac{x(mx)^3}{x^4 + (mx)^4} = \frac{m^3}{1 + m^4}, \quad x \neq 0.$$

Since this changes as m changes the limit does not exist.

(b) (10 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy^2}{x^2 + y^2}$$

exist? (Hint: How large is xy^2 relative to $x^2 + y^2$ when x and y are small?)

Solution. Let $r = \sqrt{x^2 + y^2}$. We have $|xy^2| \leq r^3$ which implies

$$\left|\frac{\sin xy^2}{x^2 + y^2}\right| \leq \frac{\sin r^3}{r^2} \leq \frac{r^3}{r^2} = r, \quad (x, y) \neq 0,$$

so the limit exists (and is 0).

4. (15 pts.) Let $(x(t), y(t), z(t)) = (t, t^2, t^3)$ for $t \neq 0$. Calculate the velocity, speed, acceleration and curvature.
Solution. Let $\mathbf{x}(t) = (x(t), y(t), z(t))$, $t \neq 0$. We have

$$\begin{aligned}
\mathbf{v}(t) &= \dot{\mathbf{x}}(t) = (1, 2t, 3t^2); \\
v(t) &= |\mathbf{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}; \\
\mathbf{a}(t) &= \dot{\mathbf{v}}(t) = (0, 2, 6t); \\
\kappa(t) &= \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)^{3/2}} = \frac{|(6t^2, -6t, 2)|}{(1 + 4t^2 + 9t^4)^{3/2}} = \sqrt{\frac{36t^4 + 36t^2 + 4}{(1 + 4t^2 + 9t^4)^3}}.
\end{aligned}$$

5. (10 pts.) Find an equation for the tangent plane to the graph of $f(x,y) = xy^2$ at $(1,2,4)$.

Solution. If $c = f(a,b)$ the formula for the equation of the tangent plane to the graph at (a,b,c) is

$$z = c + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

In this case $\frac{\partial f}{\partial x} = y^2$ and $\frac{\partial f}{\partial y} = 2xy$ which at $(1,2)$ are 4 and 4, respectively, so we get

$$z = 4 + 4(x-1) + 4(y-2) = 4x + 4y - 8.$$

6. (10 pts.) Let $f(x,y,z) = xyz + z^2$ for $(x,y,z) \in \mathbf{R}^3$, let $\mathbf{u} = \frac{1}{\sqrt{3}}(1,1,1)$ and let $\mathbf{a} = (1,2,3)$. Calculate the directional derivative

$$D_{\mathbf{u}}f(\mathbf{a}).$$

Solution. We have $\nabla f(x,y,z) = (yz, xz, xy + 2z)$ which at $(1,2,3)$ is $(6,3,8)$ so

$$D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \mathbf{u} = (6,3,8) \bullet \frac{1}{\sqrt{3}}(1,1,1) = \frac{17}{\sqrt{3}}.$$

7. (10 pts.) Calculate

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial^2}{\partial z^2}(x+y)z^2.$$

Solution. We have (changing the order to make it *really* easy)

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial^2}{\partial z^2}(x+y)z^2 = \frac{\partial^2}{\partial z^2}\frac{\partial}{\partial x}\frac{\partial}{\partial y}(x+y)z^2 = \frac{\partial^2}{\partial z^2}\frac{\partial}{\partial x}z^2 = 0.$$

8. (20 pts.) Let $f(x,y) = xy - y$ and let $C = \{(x,y) : 0 \le x \le 4 - y^2\}$. Find the maximum and minimum values of f on C and find the points in C at which f assumes these values.

Solution. The C comes in four pieces: the interior C_0; the right edge $C_1 = \{(4-t^2, t) : -2 < t < 2\}$; the left edge $\{(0,t) : -2 < t < 2\}$; and the vertices $C_3 = \{(0,2), (0,-2)\}$.

We have $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x - 1$ so the unique critical point is $(1,0)$ which is a point of C_0 at which f equals 0.

On C_1 we look for $t \in (-2,2)$ such that

$$0 = \frac{d}{dt}f(4-t^2,t) = \frac{d}{dt}3t - t^3 = 3(1-t^2)$$

which gives $t = \pm 1$ which gives the points $(3,1)$ and $(3,-1)$ at which f is 2 and -2, respectively.

On C_2 we look $t \in (-2,2)$ such that

$$0 = \frac{d}{dt}f(0,t) = \frac{d}{dt}(-t) = -1$$

which never happens.

On the points $(0,2), (0,-2)$ of C_3 the function f takes on the values -2 and 2, respectively.

We infer that the maximum value of f is 2 which is taken on precisely at the points $(3,1)$ and $(0,-2)$ and the minimum value is -2 which is taken on precisely at $(3,-1)$ and $(0,2)$.

The remaining problems are more difficult.

9. (a) (15 pts.) Suppose $\mathbf{a}$ and $\mathbf{b}$ are two nonzero vectors in $\mathbf{R}^3$ and let

$$A = \{\mathbf{x} \in \mathbf{R}^3 : \mathbf{x} \times \mathbf{a} = \mathbf{b}\}.$$

I tell you A is either a line or is empty. Give necessary and sufficient conditions for A to be a line and, in this case, give a parametric representation of this line. (**Don't** blindly write down equations. Think geometrically.)

Solution. **Case One.** $\mathbf{a} \bullet \mathbf{b} \neq 0$. Were $\mathbf{x} \in A$ we would have

$$0 = \mathbf{a} \bullet (\mathbf{x} \times \mathbf{a}) = \mathbf{a} \bullet \mathbf{b} \neq 0$$

which is impossible. Thus, in this case, A is empty.

Case Two. $\mathbf{a} \bullet \mathbf{b} = 0$. In this case, the vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ are nonzero and mutually perpendicular. It follows that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}$ is nonzero and parallel to $\mathbf{b}$, say

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \delta \mathbf{b}$$

for some nonzero scalar δ.

Given $\mathbf{x}$ in $\mathbf{R}^3$ there are scalars α, β, γ such that $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$ so

$$\mathbf{x} \times \mathbf{a} = \beta(\mathbf{b} \times \mathbf{a}) + \gamma((\mathbf{a} \times \mathbf{b}) \times \mathbf{a}) = \beta(\mathbf{b} \times \mathbf{a}) + \delta\gamma\mathbf{b}.$$

Thus $\mathbf{x} \in A$ if and only if $\beta = 0$ and $\delta\gamma = 1$ which is to say that

$$\mathbf{x} = \frac{1}{\delta}(\mathbf{a} \times \mathbf{b}) + \alpha\mathbf{a}, \quad \alpha \in \mathbf{R}.$$

(b) (15 pts.) Let

$$P = \{\mathbf{y} \in \mathbf{R}^3 : \mathbf{y} = \mathbf{x} \times (1,1,0) + (0,-2,1) \text{ for some vector } \mathbf{x} \text{ in } \mathbf{R}^3\}.$$

I tell you that P is a plane and ask you write an equation for it.

Since you know I'm right, let's work backwards. First of all, $\mathbf{x}_0 = (0,-2,1) \in P$ (let $\mathbf{x} = \mathbf{0}$. Moreover, if $\mathbf{y} \in P$ then $(\mathbf{y} - \mathbf{x}_0) \bullet (1,1,0) = 0$ so, with $\mathbf{n} = (1,1,0)$, we have

$$P \subset \{\mathbf{y} : (\mathbf{y} - \mathbf{x}_0) \bullet \mathbf{n} = 0\};$$

in particular, if $\mathbf{y} = (x,y,z)$ then $\mathbf{y} \in P$ if and only if

$$x + y = -2.$$

So there's your equation. (It's not too hard to show

$$P = \{\mathbf{y} : (\mathbf{y} - \mathbf{x}_0) \bullet \mathbf{n} = 0\};$$

just let $\mathbf{x} = t\mathbf{t} + u\mathbf{u}$, $t, u \in \mathbf{R}$, where $\mathbf{t}$ and $\mathbf{u}$ are nonparallel vectors perpendicular to $\mathbf{n}$.)

10. Suppose w, z are functions of u, v defined near $(u,v) = (0,0)$;

$$w(0,0) = 1 \quad \text{and} \quad z(0,0) = -1;$$

and

$$u^2 + v^2 + w^2 + z^2 = 0 \quad \text{and} \quad w^3 + z^3 = 0.$$

(a) (15 pts.) Calculate w_u, w_v, z_u, z_v at $(0,0)$.

Solution. Differentiating with respect to u we obtain

$$2u + 2ww_u + 2zz_u = 0 \quad \text{and} \quad 3w^2 w_u + 3z^2 z_u = 0$$

which at $(u,v) = (0,0)$ gives

$$2w_u - 2z_u = 0 \quad \text{and} \quad 3w_u + 3z_u = 0$$

which implies w_u and z_u are both 0 at $(0,0)$.

Differentiating with respect to v, we obtain

$$2v + 2ww_v + 2zz_v = 0 \quad \text{and} \quad 3w^2 w_v + 3z^2 z_v = 0$$

which at $(u,v) = (0,0)$ give

$$2w_v - 2z_v = 0 \quad \text{and} \quad 3w_v + 3z_v = 0$$

which implies w_v and z_v are both 0 at $(0,0$

(b) (30 pts.) Calculate $w_{uu}, w_{uv}, w_{vv}, z_{uu}, z_{uv}, z_{vv}$ at $(0,0)$.

Solution. We have established that

$$u + ww_u + zz_u = 0 \quad \text{and} \quad w^2 w_u + z^2 z_u = 0. \tag{1}$$

Differentiating (1) with respect to u we get

$$1 + w_u^2 + ww_{uu} + z_u^2 + zz_{uu} = 0 \quad \text{and} \quad 2ww_u^2 + w^2 w_{uu} + 2zz_u^2 + z^2 z_{uu} = 0$$

which at $(u,v) = (0,0)$ gives

$$1 + w_{uu} - z_{uu} = 0 \quad \text{and} \quad w_{uu} + z_{uu} = 0$$

which gives

$$w_{uu} = -\frac{1}{2} \quad \text{and} \quad z_{uu} = \frac{1}{2} \quad \text{at } (u,v) = (0,0).$$

Differentiating (1) with respect to v we get

$$w_v w_u + ww_{uv} + z_v z_u + zz_{uv} = 0 \quad \text{and} \quad 2ww_v w_u + w^2 w_{uv} + 2zz_v z_u + z^2 z_{uu} = 0$$

which at $(u,v) = (0,0)$ gives

$$w_{uv} - z_{uv} = 0 \quad \text{and} \quad w_{uv} + z_{uv} = 0$$

which gives

$$z_{uv} = 0 \quad \text{and} \quad w_{uv} = 0 \quad \text{at } (u,v) = (0,0).$$

We have established that

$$v + ww_v + zz_v = 0 \quad \text{and} \quad w^2 w_v + z^2 z_v = 0. \tag{2}$$

Differentiating (2) with respect to v we obtain

$$1 + w_v^2 + ww_{vv} + z_v^2 + zz_{vv} = 0 \quad \text{and} \quad 2ww_v^2 + w^2 w_{vv} + 2zz_v^2 + z^2 z_{vv}$$

which at $(u,v) = (0,0)$ gives

$$1 + w_{vv} - z_{vv} = 0 \quad \text{and} \quad w_{vv} + z_{vv} = 0$$

so, finally,

$$w_{vv} = -\frac{1}{2} \quad \text{and} \quad z_{vv} = \frac{1}{2} \quad \text{at } (u,v) = (0,0).$$

That's all folks!