Mathematics 103.07 **Test Two** **November 16, 2006**

I have neither given nor received aid in the completion of this test.
Signature:

To get full credit you must show enough work to convince me that you know what you are doing!

Problem	Possible	Grade
1	10	
2	10	
3	25	
4	25	
5	20	
6	20	
7	20	
8	35	
9	*	
10	35	
Total	200	

The average on this test was 128.08. The standard deviation was 25.16.

A student pointed out the the material in Problem 9 was not in Chapters 13 or 14. I therefore decided not to count it.

1. 10 pts. Use the method of Lagrange multipliers to determine the point(s) at which $f(x, y) = x + y^2$ attains its minimum and maximum values on the circle $x^2 + y^2 = 1$.

Solution. Set $g(x, y) = x^2 + y^2$. Then

$$\nabla f(x, y) \bullet \nabla g(x, y)^{\perp} = (1, 2y) \bullet (2x, 2y)^{\perp} = (1, 2y) \bullet (-2y, 2x) = -2y + (2y)(2x) = 2y(2x - 1)$$

which is zero when $y = 0$ or $x = 1/2$. If $y = 0$ and $g(x, y) = 1$ then $x = \pm 1$ and $f(x, y) = \pm 1$. If $x = 1/2$ and $g(x, y) = 1$ then $y = \pm\sqrt{3}/2$ and $f(x, y) = 1/2 + 3/4 = 5/4$. Thus when $g(x, y) = 1$ the function f attains its minimum value of -1 at $(-1, 0)$ and its maximum value of $5/4$ at $(1/2, \pm\sqrt{3}/2)$.

2. 10 pts. Let

$$f(x, y) = -3xy + 2x + xy^2 + 3y - y^2 \quad \text{for } (x, y) \in \mathbf{R}^2.$$

I tell you that $(1, 2)$ is a critical point of f. Apply the second derivative test to determine whether $(1, 2)$ is a relative minimum, a relative maximum, a saddle point or none of these.

Solution. We have

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -3+2y \\ -3+2y & 2x-2 \end{bmatrix}$$

so $A = f_{xx}(1,2) = 0$, $B = f_{xy}(1,2) = 1$ and $C = 0$. Since $AC - B^2 = -1$ we find that $(1,2)$ is a saddle point.

3. (a) (10 pts.) Evaluate:

$$\int_0^1 \left(\int_{x^2}^{x} x^2 y \, dy \right) dx$$

Solution.

$$\frac{1}{2}\int_0^1 x^2 y^2 |_{y=x^2}^{y=x} \, dx = \frac{1}{2}\int_0^1 x^4 - x^6 \, dx = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = -\frac{1}{35}.$$

(b) (15 pts.)

$$\int_0^1 \left(\int_x^{2x} \left(\int_{xy}^{2xy} xyz \, dz \right) dy \right) dx.$$

Solution.

$$\begin{aligned}
&\int_0^1 \left(\int_x^{2x} \frac{xyz^2}{2} |_{z=xy}^{z=2xy} \, dy \right) dx \\
&= \frac{3}{2}\int_0^1 \left(\int_x^{2x} x^3 y^3 \, dy \right) dx \\
&= \frac{3}{2}\int_0^1 \frac{x^3 y^4}{4} |_{y=x}^{y=2x} \, dx \\
&= \frac{3}{2}\frac{15}{4}\int_0^1 x^7 \, dx \\
&= \frac{3}{2}\frac{15}{4}\frac{1}{8} \\
&= \frac{45}{64}.
\end{aligned}$$

4. Represent as an iterated integral:

(a) (10 pts.)

$$\int\int_T xy \, dA$$

where T is the triangle with vertices at $(0,0), (1,0), (1,1)$.

(b) (15 pts).

$$\int\int_R xy \, dA$$

where R is the bounded region between the parabolas $y = 3 - x^2$ and $y = x^2$.

Solution to (a).

$$\int_0^1 \left(\int_0^x xy \, dy \right) dx = \frac{1}{8}.$$

Solution to (b). The projection of R on the x-axis is the interval with endpoints at the solutions of $3 - x^2 = x^2$ or $x = \pm\sqrt{\frac{3}{2}}$. Thus the integral equals

$$\int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} \left(\int_{x^2}^{3-x^2} xy\, dy \right) dx = 0.$$

(Of course it's obvious it's zero because of the symmetry of R about the y-axis.)

5. 20 pts. Let R be the set of those $(x, y) \in \mathbf{R}^2$ such that $x > 0, y > 0$,

$$1 < (xy)^2 < 3 \quad \text{and} \quad 2 < y/x < 4.$$

Use the Change of Variables Formula for Multiple Integrals to compute the area of R.
Solution. Let $u = (xy)^2$ and let $v = y/x$ for $x > 0$ and $y > 0$ and note that $(x, y) \to (u, v)$ is one-to-one. We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)} \right)^{-1} = (u_x v_y - u_y v_x)^{-1} = \left((2xy^2)(1/x) - (2x^2y)(-y/x^2) \right)^{-1} = (4y^2)^{-1} = \frac{1}{4\sqrt{uv}}$$

so the area of R equals

$$\int\int_{1<u<3,\ 2<v<4} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv = \int_1^3 \left(\int_2^4 \frac{1}{\sqrt{uv}}\, dv \right) du = 2\ln(2)(\sqrt{3} - 1).$$

6. 20 pts. Use polar coordinates to represent the volume of the solid bounded by the paraboloids $z = 12 - 2x^2 - y^2$ and $z = x^2 + 2y^2$.

Solution. The projection R of the solid on the xy-plane is bounded by the solution set of $12 - 2x^2 - y^2 = x^2 + 2y^2$ or $x^2 + y^2 = 2$. Thus the volume of the solid equals

$$\int\int_R (12 - 2x^2 - y^2) - (x^2 + 2y^2)\, dxdy = \int\int_R 12 - 3(x^2 + y^2)\, dxdy = \int_0^2 \left(\int_0^{2\pi} 12 - 3r^2\, r\, dr \right) d\theta = 32\pi.$$

7. 20 pts. Let S be the solid consisting of those points in $\mathbf{R}^3$ such that $x^2+y^2+z^2 < 3^2$ and $|z| \le \sqrt{x^2 + y^2}$. Use spherical coordinates to represent the volume of S as an iterated integral.

Solution. The key point is that ϕ varies from $\pi/4$ to $3\pi/4$. The desired volume equals

$$\int\int\int_{0<\rho<3,\ \pi/4<\phi<3\pi/4,\ 0<\theta<2\pi} \rho^2 \sin\phi\, d\rho d\phi d\theta = \int_0^{2\pi} \left(\int_{\pi/4}^{3\pi/4} \left(\int_0^3 \rho^2 \sin\phi\, d\rho \right) d\phi \right) d\theta = 18\sqrt{2}\pi.$$

8. Let T be the tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 2, 0), (0, 0, 3)$.

(a) (15 pts.) Compute the volume of T.

(b) (20 pts.) Compute the area of the boundary of T.
(Hint: Determine a, b, c such that the vertices $(1, 0, 0), (0, 2, 0), (0, 0, 3)$ satisfy $ax + by + cz = 1$.)

Solution. The vertices $(1, 0, 0), (0, 2, 0), (0, 0, 3)$ satisfy

$$x + \frac{y}{2} + \frac{z}{3} = 1$$

and so the face U of the tetrahedron containing these three points is in the graph of $z = 3(1 - x - \frac{y}{2})$. Let R be the triangle containing the points $(0,0), (1,0), (0,2)$. The volume of T is

$$\int\int_R 3\left(1 - x - \frac{y}{2}\right)\, dxdy = \int_0^1 \left(\int_0^{2x-2} 3\left(1 - x - \frac{y}{2}\right)\, dy\right) dx = 1.$$

The area of R is 1. The area of the triangle Q containing the vertices $(0,0,0), (1,0,0), (0,0,3)$ is $\frac{3}{2}$. The area of the triangle P containing the vertices $(0,0,0), (0,2,0), (0,0,3)$ is 3. Since

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + (-3)^2 + (-3/2)^2} = \frac{7}{2}$$

the area of U is

$$\int\int_R \frac{7}{2}\, dxdy = \frac{7}{2}.$$

Hence the area of the boundary of T is

$$1 + \frac{3}{2} + 3 + \frac{7}{2} = 9.$$

9. 15 pts. Express as a definite integral:

$$\int_C (x + y + z)\, ds$$

where C is the curve $\{(\cos\theta, \sin\theta, \theta) : 0 \leq \theta \leq 4\pi\}$. (Note that C is a segment of a helix.)

Solution. Let $x = \cos\theta, y \sin\theta, z = \theta,\ 0 \leq \theta \leq 4\pi$. Then

$$ds = \sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2 + \frac{dz}{\theta}^2} = \sqrt{2}$$

so

$$\int_C (x + y + z)\, ds = \int_0^4 \pi(\cos\theta + \sin\theta + \theta)\sqrt{2}\, d\theta = \sqrt{2}\pi(9 + \sin 4 + \cos 4).$$

10. Let

$$\mathbf{r}(u, v) = \left(uv, u^2, v^2\right), \quad \text{for } (u, v) \in \mathbf{R}^2,$$

and let

$$T = \{(u, v) \in \mathbf{R}^2 : 0 < u < v < 1\}$$

and let S be the surface

$$\{\mathbf{r}(u, v) : (u, v) \in T\}.$$

(a) (15 pts.) Express the area of S as an iterated integral.

Solution. We have

$$\mathbf{r}_u = (v, 2u, 0), \quad \mathbf{r}_v = (u, 0, 2v)$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = (v\mathbf{i} + 2u\mathbf{j}) \times (u\mathbf{i} + 2v\mathbf{k}) = (4uv, -2v^2, -2u^2)$$

the length of which is

$$\sqrt{16u^2v^2 + 4v^4 + 4u^4}.$$

Thus the desired area is

$$\int\int_{0<u<1,\ u<v<1} \sqrt{16u^2v^2 + 4v^4 + 4u^2}\, dudv = \int_0^1 \left(\int_u^1 \sqrt{16u^2v^2 + 4v^4 + 4u^2}\, dv \right) du.$$

Maple couldn't do it and I didn't try.

(b) (20 pts.) Show that $\mathbf{r}$ is one-to-one on T.

Solution. Suppose $(u_i, v_i) \in T$ and $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$. Then

$$u_1 v_1 = u_2 v_2, \quad u_1^2 = u_2^2, \quad v_1^2 = v_2^2.$$

Because $u_1 > 0$ and $u_2 > 0$ we infer that $u_1 = u_2$ and because $v_1 > 0$ and $v_2 > 0$ we infer that $v_1 = v_2$.

That's all folks!