Curves in $\mathbb{R}^n$

1. Limits, continuity and differentiation.

Throughout this section, I is an interval, $a \in I$ and

$$\mathbf{r} = (r_1, \dots, r_n) : I \to \mathbb{R}^n.$$

In 12.5 in the book $n = 3$. Often, for the sake of brevity, we will say $\mathbf{r}$ is a **curve**. Notice the difference between the range of $\mathbf{r}$ (which the book calls the track) and $\mathbf{r}$ itself.

Definition 1.1. Suppose $\mathbf{l} = (l_1, \dots, l_n) \in \mathbb{R}^n$. We say $\mathbf{r}(t)$ approaches $\mathbf{l}$ as t approaches a and write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{l} \tag{1}$$

if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$t \in I \text{ and } 0 < |t - a| < \delta \;\Rightarrow\; |\mathbf{r}(t) - \mathbf{l}| < \epsilon.$$

Theorem 1.1. Suppose $\mathbf{l} = (l_1, \dots, l_n) \in \mathbb{R}^n$. Then (1) holds if and only if

$$\lim_{t \to a} r_i(t) = l_i \quad \text{for } i \in \{1, \dots, n\}.$$

Definition 1.2. We say $\mathbf{r}$ is **differentiable at** a if a is an interior point of I and there is $\mathbf{r}'(a) \in \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{\mathbf{r}(a + h) - \mathbf{r}(a)}{h} = \mathbf{r}'(a). \tag{2}$$

Theorem 1.2. $\mathbf{r}$ is differentiable at $\mathbf{a}$ if and only if r_i is differentiable at a for each $i \in \{1, \dots, n\}$ in which case we have

$$\mathbf{r}'(a) = (r_1'(a), \dots, r_n'(a)).$$

2. Limit and differentiation rules.

In 12.5 there ought to be limit rules following the pattern of Theorem 2. Let me illustrate by example. Throughout this section I is an open interval, $a \in I$ and

$$\mathbf{u}, \mathbf{v} : I \to \mathbb{R}^3.$$

Theorem 2.1. Suppose $\lim_{t \to a} \mathbf{u}(t) = \mathbf{b}$ and $\lim_{t \to a} \mathbf{v}(t) = \mathbf{c}$. Then

$$\lim_{t \to a} (\mathbf{u} \times \mathbf{v})(t) = \mathbf{b} \times \mathbf{c}.$$

Proof. Suppose $t \in I$. Then

$$\begin{aligned}
&|\mathbf{u} \times \mathbf{v})(t) - \mathbf{u} \times \mathbf{v})(a)| \\
&\qquad = |(\mathbf{u}(t) - \mathbf{u}(a)) \times \mathbf{v}(a) + \mathbf{u}(t) \times (\mathbf{v}(t) - \mathbf{v}(a))| \\
&\qquad \le |(\mathbf{u}(t) - \mathbf{u}(a)) \times \mathbf{v}(a)| + |\mathbf{u}(t) \times (\mathbf{v}(t) - \mathbf{v}(a))| \\
&\qquad \le |(\mathbf{u}(t) - \mathbf{u}(a)| \, |\mathbf{v}(a)| + |\mathbf{u}(t)|, |\mathbf{v}(t) - \mathbf{v}(a))|;
\end{aligned}$$

this last quantity approaches 0 as t approaches a by limit rules from one variable calculus; here we have used the triangle inequality and the fact that the length of a cross product does not exceed the product of the length of the factors. □

Theorem 2.2. Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable at a. Then $\mathbf{u}\times\mathbf{v}$ is differentiable at a and

$$(\mathbf{u} \times \mathbf{v})'(a) = \mathbf{u}'(a) \times \mathbf{v}(a) + \mathbf{u}(a) \times \mathbf{v}'(a).$$

Proof. Suppose $h \in \mathbb{R}$ and $a + h \in I$. Then

$$\begin{aligned}&\frac{1}{h}\left((\mathbf{u} \times \mathbf{v})(a+h) - (\mathbf{u} \times \mathbf{v})(a)\right)\\ &\quad = \left(\frac{\mathbf{u}(a+h) - \mathbf{u}(a)}{h}\right) \times \mathbf{v}(a) + \mathbf{u}(a+h) \times \left(\frac{\mathbf{v}(a+h) - \mathbf{v}(a)}{h}\right);\end{aligned}$$

now apply limit rules to obtain the desired result. □

3. Velocity, acceleration and speed.

Definition 3.1. Suppose $\mathbf{r}$ is a curve. Then

$\mathbf{r}'$ is its **velocity**,

$\mathbf{r}''$ is its **acceleration**

and

$|\mathbf{r}'|$ is its **speed.**

(So velocity and acceleration are vectors and speed is a scalar. Forgetting this leads to all sorts of confusion.)

4. Integration.

Suppose $\mathbf{r}$ is a curve. Then its **integral**

$$\int_a^b \mathbf{r}(t)\,dt$$

can be defined using Riemann sums in the same was one defines the integral of a scalar. Note the stuff on pages 809-811.

5. Projectile motion.

Suppose $\mathbf{r}$ is the path of a projectile with mass m which is subject to the force $-g\mathbf{k}$ where g is the gravitational constant for the Earth's surface in units consistent with those of m. Then Newton's Second Law of Motion says

$$(m\mathbf{r}')' = -g\mathbf{k}.$$

If the mass is constant this becomes

$$m\mathbf{r}'' = -g\mathbf{k}.$$

Let $t_0 \in I$, let

$$\mathbf{r}_0 = \mathbf{r}(t_0) \quad \text{and let} \quad \mathbf{v}_0 = \mathbf{r}'(t_0).$$

(That is, $\mathbf{r}_0$ and $\mathbf{v}_0$ are the **initial position and velocity**, respectively. Integrating from t_0 to t we obtain

$$m(\mathbf{r}'(t) - \mathbf{v}_0) = -g(t - t_0)\mathbf{k}$$

so

$$\mathbf{r}'(t) = \mathbf{v}_0 - \frac{g}{m}(t - t_0)\mathbf{k}.$$

Integrating one more time from t_0 to t we obtain

$$\mathbf{r}(t) - \mathbf{r}_0 = (t - t_0)\mathbf{v}_0 - \frac{g}{2m}(t - t_0)^2\mathbf{k}$$

so

$$\mathbf{r}(t) = \mathbf{r}_0 + (t - t_0)\mathbf{v}_0 - \frac{g}{2m}(t - t_0)^2\mathbf{k}.$$

In particular, the range of $\mathbf{r}$ lies in any plane containing the initial position, the initial velocity and $\mathbf{k}$.

6. Problem 62 on page 815.

As written it makes no sense. What they probably mean is that if in Newton's Second Law

$$\mathbf{F} = m\mathbf{a}$$

(constant mass) where we are moving in $\mathbb{R}^3$ we have

$$\mathbf{F}||\mathbf{r}$$

then the range (or track in the book) of $\mathbf{r}$ lies in a plane. This is very interesting and useful. It's why, for example, the planetary motion, in the two body version, lies in a plane containing the Sun.