1. Differentiability, differentials, linear approximation and all that stuff.

Suppose n is a positive integer, $A \subset \mathbb{R}^n$,

$$f : A \to \mathbb{R}$$

and $\mathbf{a} \in A$.

Definition 1.1. We say f is **differentiable at a** if there exists a vector $\mathbf{w} \in \mathbb{R}^n$ such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|f(\mathbf{x}) - f(\mathbf{a}) - (\mathbf{x}-\mathbf{a}) \bullet \mathbf{w}|}{|\mathbf{x}-\mathbf{a}|} = 0. \tag{1}$$

The vector $\mathbf{w}$ is immediately seen to be unique; it is called the **gradient of** f **at a** and is written

$$\nabla f(\mathbf{a}).$$

If we let

$$A(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x}-\mathbf{a}) \bullet \nabla f(\mathbf{a}) \quad \text{and} \quad \mathbf{e}(\mathbf{x}) = f(\mathbf{x}) - A(\mathbf{x})$$

for $\mathbf{x} \in A$ we find that (1) is equivalent to

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|e(\mathbf{x})|}{|\mathbf{x}-\mathbf{a}|} = 0. \tag{2}$$

One calls the function A the **standard affine approximation to** f **at a.** The difference $e = f - A$ is the error in using A to approximate f.

Theorem 1.1. Suppose for some $r > 0$ the function f has partial derivatives on $A \cap \mathbf{U}(\mathbf{a}, r)$ which are continuous at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$ and

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{a})\right).$$

Proof. We'll do the case $n = 2$; it will be obvious how to handle the general case.

Let $\mathbf{a} = (a, b)$ and suppose $(x, y) \in A \cap \mathbf{U}((a, b), r)$. We consider the case $a < x$ and $b < y$ and leave to the reader the simple task of dealing with the other cases. Let

$$\mathbf{w} = \left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b)\right).$$

Suppose $(x, y) \in A \cap \mathbf{U}((a, b), r)$. Applying the Mean Value Theorem twice we obtain $\xi \in (a, x)$ and $\eta \in (b, y)$ such that

$$\begin{aligned}
f(x, y) - f(a, b) &= [f(x, b) - f(a, b)] + [f(x, y) - f(x, b)] \\
&= \frac{\partial f}{\partial x}(\xi, b)(x - a) + \frac{\partial f}{\partial y}(x, \eta)(y - b) \\
&= \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \\
&+ \left(\frac{\partial f}{\partial x}(\xi, b) - \frac{\partial f}{\partial x}(a, b)\right)(x - a) + \left(\frac{\partial f}{\partial y}(x, \eta) - \frac{\partial f}{\partial y}(a, b)\right)(y - b) \\
&= \mathbf{w} \bullet ((x, y) - (a, b)) + \rho(x, y)
\end{aligned}$$

where we have set

$$\rho(x, y) = \left(\frac{\partial f}{\partial x}(\xi, b) - \frac{\partial f}{\partial x}(a, b)\right)(x - a) + \left(\frac{\partial f}{\partial y}(x, \eta) - \frac{\partial f}{\partial y}(a, b)\right)(y - b).$$

Since

$$\lim_{(x,y)\to(a,b)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(a,b) \quad \text{and} \quad \lim_{(x,y)\to(a,b)} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(a,b)$$

we find that

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - f(a,b) - ((x,y)-(a,b)) \bullet \mathbf{w}|}{|(x,y)-(a,b)|} = 0$$

so that f is differentiable at (a,b) and $\nabla f(a,b) = \mathbf{w}$. □

Definition 1.2. Suppose f is differentiable at $\mathbf{a}$. Then the **differential**

$$df(\mathbf{a})$$

of f **at** $\mathbf{a}$ is the function from $\mathbb{R}^n$ into $\mathbb{R}$ such that

$$df(\mathbf{a})(\mathbf{v}) = \mathbf{v} \bullet \nabla f(\mathbf{a}).$$

Remark 1.1. It follows immediately that if f is differentiable at $\mathbf{a}$ then

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|f(\mathbf{x}) - (f(\mathbf{a}) + df(\mathbf{a})(\mathbf{x}-\mathbf{a}))|}{|\mathbf{x}-\mathbf{a}|} = 0.$$

This is the essence of Section 13.6 in the book

2. Further analysis of the errror.

Suppose $A \subset \mathbb{R}^2$,

$$f : A \to \mathbb{R}$$

and $(a,b), \in \textbf{int}\, A$. Suppose (x,y) is such that

$$S = \{(1-t)(a,b) + t(x,y) : 0 \le t \le 1\} \subset \textbf{int}\, A$$

and that the first partial derivatives of f exist and are differentiable at each point of S.

Let

$$A(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

How big is

$$f(x,y) - A(x,y)?$$

Let

$$Q(x,y,u,v) = \frac{\partial^2 f}{\partial x^2}(x,y)\frac{u^2}{2} + \frac{\partial^2 f}{\partial x \partial y}(x,y)uv + + \frac{\partial^2 f}{\partial y^2}(x,y)\frac{v^2}{2}.$$

Using the Lagrange form of the remainder and the Chain Rule (See if you can carry this out after we study the Chair Rule!) one find that There is $t \in [0,1]$ such that

$$f(x,y) - A(x,y) = Q(a + t(x-a), b + t(y-b), x-a, y-b).$$