Green's Theorem.

Suppose

(1)

$$\mathbf{r} : [a,b] \times [0,1] \to \mathbf{R}^2$$

is one to one and continuously differentiable and

$$\mathbf{det} \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} > 0$$

where x, y are such that $\mathbf{r}(s,t) = (x(s,t), y(s,t))$;

(2)

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

is a continuously differentiable vector field defined on D where D is the range of $\mathbf{r}$ and

(3) C is the closed curve in $\mathbf{R}^2$ obtained by applying $\mathbf{r}$ to the boundary of $[a,b] \times [0,1]$ traversed in the counterclockwise sense: First apply $\mathbf{r}$ to the points $(s,0)$ as s increases from a to b; then apply $\mathbf{r}$ to the points (b,t) as t increases from 0 to 1; then apply $\mathbf{r}$ to the points $(s,1)$ as s decreases from b to a; finally, apply $\mathbf{r}$ to the points (a,t) as t decreases from 1 to 0.

Then

(4) $$\iint_D Q_x - P_y\,dA = \int_C \mathbf{F} \bullet \mathbf{T}\,ds \Big(= \int_C P\,dx + Q\,dy. \Big)$$

Remark. The argument we give will also work to give Stokes' Theorem, but more computation is required. Try it and see what I mean.

Proof. Let L be the right hand side of (4) let R be the left hand side of (4). Set

$$I_1 = \int_a^b P(x(s,t))\frac{\partial x}{\partial s}(s,t) + Q(x(s,t))\frac{\partial y}{\partial s}(s,t)\Big|_{t=0}^{t=1} ds$$

and set

$$I_2 = \int_0^1 P(x(s,t))\frac{\partial x}{\partial t}(s,t) + Q(x(s,t))\frac{\partial y}{\partial t}(s,t)\Big|_{s=a}^{s=b} dt.$$

Note that

$$R = I_2 - I_1.$$

Using the Fundamental Theorem of Calculus we obtain

$$I_1 = \int_a^b \int_0^1 \frac{\partial}{\partial t}\big[P(x(s,t))\frac{\partial x}{\partial s}(s,t) + Q(x(s,t))\frac{\partial y}{\partial s}(s,t)\big]\,dt\,ds$$

and

$$I_2 = \int_0^1 \int_a^b \frac{\partial}{\partial s}\big[P(x(s,t))\frac{\partial x}{\partial t}(s,t) + Q(x(s,t))\frac{\partial y}{\partial t}(s,t)\big]\,ds\,dt.$$

Now

$$[Px_s + Qy_s]_t = P_x x_t x_s + P_y y_t x_s + P x_{st} + Q_x x_t y_s + Q_y y_t y_s + Q y_{st}$$

and

$$[Px_t + Qy_t]_s = P_x x_s x_t + P_y y_s x_t + P x_{ts} + Q_x x_s y_t + Q_y y_s y_t + Q y_{ts}.$$

Using the equality of mixed partial derivatives and the fact that $\int_a^b \int_0^1 = \int_0^1 \int_a^b$ we find that

$$R = \iint_{[a,b]\times[0,1]} \big[Q_x(x(s,t),y(s,t)) - P_y(x(s,t),y(s,t))\big]\big[x_s(s,t)y_t(s,t) - x_t(s,t)y_s(s,t)\big]\,dA.$$

The proof may now be completed using the change of variables formula for double integrals. □