1. Integration.

Definition 1.1. We say $f : \mathbb{R}^2 \to \mathbb{R}$ is **admissible** if $|f|$ is bounded and $\{(x, y) \in \mathbb{R}^2 : f(x, y) \neq 0\}$ is bounded.

Definition 1.2. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is admissible and $0 < \delta < \infty$. We let

$$\mathcal{R}_\delta(f)$$

be the set of sums

$$S = \sum_{i=1}^{M} \sum_{j=1}^{N} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1})$$

where

(i) M and N are positive integers;
(ii) $x_0 \leq x_1 \leq \cdots \leq x_M$ and $y_0 \leq y_1 \leq \cdots \leq y_N$;
(iii) $x_i - x_{i-1} \leq \delta$ for $i = 1, \ldots, M$ and $y_j - y_{j-1} \leq \delta$ for $j = 1, \ldots, N$;
(iv) $x_{i-1} \leq \xi_i \leq x_i$ for $i = 1, \ldots, M$ and $y_{j-1} \leq \eta_j \leq y_j$ for $j = 1, \ldots, N$;
(v) $\{(x, y) \in \mathbb{R}^2 : f(x, y) \neq 0\} \subset [x_0, x_M] \times [y_0, y_N]$.

We say f **is Riemann integrable** if there is $I \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$S \in \mathcal{R}_\delta(f) \;\Rightarrow\; |S - I| < \epsilon.$$

It is evident that such an I is unique; we denote it by

$$\int f$$

and call it the **Riemann integral of f.**

Theorem 1.1. Suppose $a, b \in \mathbb{R}$ and f, g are Riemann integrable. Then $af + bg$ is Riemann integrable and

$$\int af + bg = a \int f + b \int g.$$

Theorem 1.2. Suppose f, g are Riemann integrable and $f \leq g$. Then

$$\int f \leq \int g.$$

Theorem 1.3. Suppose $f_1, f_2, \ldots, f_n, \ldots$ is a sequence of Riemann integrable functions and $F : \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\lim_{n\to\infty} \sup |F - f_n| = 0.$$

Then F is Riemann integrable and

$$\lim_{n\to\infty} \int f_n = \int F.$$

1.1. **Indicator functions.**

Definition 1.3. Suppose $A \subset \mathbb{R}^2$ we let

$$1_A : \mathbb{R}^2 \to \{0,1\}$$

be such that

$$1_A(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A, \\ 0 & \text{if } (x,y) \notin A. \end{cases}$$

We call 1_A the **indicator function of** A**.** Note that 1_A is admissible if and only if A is bounded.

Definition 1.4. Suppose $A \subset \mathbb{R}^2$. We say A **has Jordan content** if A is bounded and 1_A is Riemann integrable in which case we let

$$|A| = \int 1_A$$

and we call this nonnegative real number the **Jordan content of** A. Note that the empty set has zero Jordan content.

Theorem 1.4. Suppose $-\infty < a \le b < \infty$ and $-\infty < c \le d < \infty$. Then $R = [a,b] \times [c,d]$ has Jordan content and

$$|R| = (b-a)(d-c).$$

Theorem 1.5. A bounded subset A of $\mathbb{R}^2$ has zero Jordan content if and only if for each $\epsilon > 0$ there is a finite sequence $R_1, \ldots, R_N$ of bounded rectangles such that

$$A \subset \cup_{i=1}^N R_i \quad \text{and} \quad \sum_{i=1}^N |R_i| < \epsilon.$$

Theorem 1.6. A bounded subset A of $\mathbb{R}^2$ has Jordan content if and only if its boundary has zero Jordan content.

Theorem 1.7. Suppose A and B have Jordan content. Then $A \cap B$, $A \cup B$ and $A \sim B$ have Jordan content and

$$|A \cup B| + |A \cap B| = |A| + |B|.$$

1.2. **A simple but useful generalization.**

Definition 1.5. Suppose $A \subset \mathbb{R}^2$ and $f : A \to \mathbb{R}$. We say f **is Riemann integrable (over** A**)** if f_A is Riemann integrable where $f_A : \mathbb{R}^2 \to \mathbb{R}$ is such that

$$f_A(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in A, \\ 0 & \text{if } (x,y) \notin A, \end{cases}$$

in which case we let

$$\int_A f \quad \text{equal} \int f_A.$$

1.3. **A sufficient condition for Riemann integrability.**

Theorem 1.8. Suppose A is a bounded subset of $\mathbb{R}^2$ with Jordan content and $f : A \to \mathbb{R}$. Then f is Riemann integrable over A if and only if f is bounded and the set of discontinuities of f is a set of zero Jordan content.

1.4. **Iterated integrals.**

Theorem 1.9. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is Riemann integrable and $\mathbf{u}$ and $\mathbf{v}$ are mutually perpendicular unit vectors. Then

$$\int f = \int \left(\int f(u\mathbf{u} + v\mathbf{v})\, du \right) dv.$$

Remark 1.1. In particular, if $\mathbf{u} = \mathbf{i} = (1, 0)$ and $\mathbf{v} = \mathbf{j} = (0, 1)$ we have

$$\int f = \int \left(\int f(x, y)\, dx \right) dy$$

and if $\mathbf{u} = \mathbf{j} = (0, 1)$ and $\mathbf{v} = \mathbf{i} = (1, 0)$ we have

$$\int f = \int \left(\int f(x, y)\, dy \right) dx.$$

Remark 1.2. This is not quite right but I don't want to go into it because (i) it's gets technical; (ii) it's really not an issue in nearly every application; and (iii) the trouble is caused by the Rieman integral and and may be avoided by using the Lebesgue integral.

Remark 1.3. In particular, if $\mathbf{u} = \mathbf{i} = (1, 0)$ and $\mathbf{v} = \mathbf{j} = (0, 1)$ we have

$$\int f = \int \left(\int f(x, y)\, dx \right) dy$$

and if $\mathbf{u} = \mathbf{j} = (0, 1)$ and $\mathbf{v} = \mathbf{i} = (1, 0)$ we have

$$\int f = \int \left(\int f(x, y)\, dy \right) dx.$$