1. The general transformation formula.

Theorem 1.1. Suppose

(i) k and n are integers and $1 \le k \le n$;

(ii) A is a closed and bounded subset of $\mathbb{R}^k$;

(iii) $\mathbf{F} : A \to \mathbb{R}^n$ and

  (a) $\mathbf{F}$ is one-one;

  (b) $\mathbf{F}$ has a continuously differentiable extension to an open set containing A;

(iv) $g : \mathbf{F}[A] \to \mathbb{R}$ and g is continuous.

Then

$$\int_{\mathbf{F}[A]} g = \int_A (g \circ \mathbf{F})\, \mathbf{JF}$$

where, for each $\mathbf{x} \in A$,

$$\mathbf{JF}(\mathbf{x})$$

is the square root of the sum of the squares of the determinants of the $\binom{n}{k}$ k by k submatrices of the n by k matrix whose (i, j) entry is

$$\partial_j F_i(\mathbf{x}), \quad i = 1, \dots, n, \quad j = 1, \dots, k.$$

Remark 1.1. In Math 103 we deal with the following cases:

$$\begin{aligned} k &= 1, \quad n = 1, 2, 3; \\ k &= 2, \quad n = 2, 3; \\ k &= 3, \quad n = 3; \end{aligned}$$

Definition 1.1. A subset T of $\mathbb{R}^n$ is a **k-dimensional simplex** if there are

$$\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k,$$

the **vertices of** T, such that T is the set of sums

$$\sum_{i=0}^{k} c_i \mathbf{v}_i$$

corresponding to c_i with $0 \le c_i \le 1$ and $\sum_{i=0} c_i = 1$. We let

$$||T||_k$$

be the square root of the determinant of the k by k matrix whose (i, j) entry is

$$(\mathbf{v}_i - \mathbf{v}_0) \bullet (\mathbf{v}_j - \mathbf{v}_0).$$

One easily verifies that this number is independent of the ordering of the vertices and that

$$||T||_k \le (\mathbf{diam}\, T)^k.$$

For example, if $k = 1$ then

$$||T||_1 = |\mathbf{v}_1 - \mathbf{v}_0| \quad \text{and} \quad \mathbf{JF}(x) = |\mathbf{F}'|(x);$$

if $k = 2$ and $n = 2$ then

$$||T||_2 = |[\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0]| \quad \text{and} \quad \mathbf{JF}(\mathbf{x}) = |[\partial_1 \mathbf{F}, \partial_2 \mathbf{F}]|(\mathbf{x});$$

if $k = 2$ and $n = 3$ then

$$||T||_2 = |(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)| \quad \text{and} \quad \mathbf{JF}(\mathbf{x}) = |\partial_1 \mathbf{F} \times \partial_2 \mathbf{F}|(\mathbf{x})$$

and if $k = 3$ and $n = 3$ then

$$||T||_3 = |[\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \mathbf{v}_3 - \mathbf{v}_0]| \quad \text{and} \quad \mathbf{JF}(\mathbf{x}) = |[\partial_1 \mathbf{F}, \partial_2 \mathbf{F}, \partial_3 \mathbf{F}]|(\mathbf{x}).$$

Suppose A, $\mathbf{F}$ and g are as above. The integral

$$\int_{\mathbf{F}[A]} g$$

is defined as follows. Suppose $\mathcal{T}$ is a triangulation of a subset of the domain of $\mathbf{F}$ which includes A and each of whose simplices of which intersect A. For each $T \in \mathcal{T}$ let

$$T_{\mathbf{F}}$$

be the simplex, possibly degenerate, in $\mathbb{R}^n$ whose vertices are the points $\mathbf{F}(\mathbf{v})$ where $\mathbf{v}$ is a vertex of T. For each $T \in \mathcal{T}$ let

$$\xi_T$$

be a point of $T \cap A$.

Suppose

$$0 < \delta < 1.$$

Then among triangulations $\mathcal{T}$ as above with

$$\min_{T \in \mathcal{T}} \frac{||T||_k}{(\mathbf{diam}\, T)^k} \geq \delta$$

the sums

$$\sum_{T \in \mathcal{T}} g(\mathbf{F}(\xi_T))\, ||T_{\mathbf{F}}||_k$$

converge to

$$\int_{\mathbf{F}[A]} g \quad \text{as} \quad \max_{T \in \mathcal{T}} \mathbf{diam}\, T \to 0.$$

The basic ingredient in the proof of the above integration formula is Inverse Function Theorem and the fact that if $\mathbf{a} \in A$, $0 < \delta < 1$ and T is a simplex in $\mathbb{R}^k$ such that

$$T \subset \mathbf{dmn}\, \mathbf{F} \quad \text{and} \quad \frac{||\mathbf{T}||_{\mathbf{k}}}{(\mathbf{diam}\, \mathbf{T})^{\mathbf{k}}} \geq \delta$$

then

$$\frac{||T||_{\mathbf{F}}}{||T||_k} \to \mathbf{JF}(\mathbf{a}) \quad \text{as} \quad \mathbf{diam}\, T \to 0.$$

2. Invariance under reparameterization.

Theorem 2.1. Suppose

(i) k and n are integers and $1 \leq k \leq n$;
(ii) A_i, $i = 1, 2$, are a closed and bounded subset of $\mathbb{R}^k$;
(iii) $\mathbf{F}_i : A \to \mathbb{R}^n$, $i = 1, 2$, and
  (a) $\mathbf{F}_i$ is one-one;
  (b) $\mathbf{F}_i$ has a continuously differentiable extension to an open set containing A;
  (c) $F_1[A_1] = B = F_2[A_2]$;
(iv) $g : B \to \mathbb{R}$ and g is continuous.

Then

$$\int_B g = \int_{\mathbf{F}_i[A_i]} g, \quad i = 1, 2.$$

The proof reduced to the product rule for determinants.