n-coordinates.

Let S be a set. We say $\mathbf{y}$ is an **m-variable** on S if $\mathbf{y} : S \to \mathbf{R}^m$. We say $\mathbf{x}$ is an **n-coordinate on** S if $\mathbf{x}$ is an n-variable on S, the range of $\mathbf{x}$ is an open subset of $\mathbf{R}^n$ and $\mathbf{x}$ is one to one.

Suppose $\mathbf{y}$ is an m-variable on S and $\mathbf{x}$ is an n-coordinate on S. Note that $\mathbf{y} \circ \mathbf{x}^{-1}$ is a function whose domain is the range of $\mathbf{x}$ and whose range is the range of $\mathbf{y}$. Evidently,

$$\mathbf{y} = (\mathbf{y} \circ \mathbf{x}^{-1}) \circ \mathbf{x}.$$

That is, the function $\mathbf{y} \circ \mathbf{x}^{-1}$ is what you do to $\mathbf{x}$ to get $\mathbf{y}$. We say

$$\mathbf{y} = \mathbf{b} \quad \textbf{when} \quad \mathbf{x} = \mathbf{a}$$

and write

$$\mathbf{y}|_{\mathbf{x}=\mathbf{a}} = \mathbf{b}$$

if $\mathbf{a}$ is in the range of $\mathbf{x}$ and $\mathbf{y}(\mathbf{x}^{-1}(\mathbf{a})) = \mathbf{b}$. For each $j = 1, \ldots, n$ we set

$$\frac{\partial \mathbf{y}}{\partial x_j} = \partial_j(\mathbf{y} \circ \mathbf{x}^{-1}) \circ \mathbf{x}$$

and call $\frac{\partial \mathbf{y}}{\partial x_j}$ the **partial derivative of $\mathbf{y}$ with respect to** x_j; note that $\frac{\partial \mathbf{y}}{\partial x_j}$ is a $\mathbf{R}^m$ valued function whose domain is a subset of S. *This notation is ambiguous! Do you see why?*

Suppose $s \in S$. We say $\mathbf{y}$ **differentiable with respect to $\mathbf{x}$ at** s if $\mathbf{y} \circ \mathbf{x}^{-1}$ is differentiable at $\mathbf{x}(s)$. We say $\mathbf{y}$ **differentiable with respect to** $\mathbf{x}$ if $\mathbf{y} \circ \mathbf{x}^{-1}$ is differentiable at $\mathbf{x}(s)$ at each s in S.

We have the following.

Differentiating with respect to more that one n-coordinate. Suppose $\mathbf{x}$ and $\mathbf{t}$ are coordinates on the set S and $\mathbf{y}$ is an m-variable on S. Suppose $\mathbf{y}$ is differentiable with respect to $\mathbf{x}$ and $\mathbf{x}$ is differentiable with respect to $\mathbf{t}$.

Then $\mathbf{y}$ is differentiable with respect to $\mathbf{t}$ and

$$\frac{\partial \mathbf{y}}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial t_j} \frac{\partial \mathbf{y}}{\partial x_i} \qquad \text{for each } j = 1, \ldots, n.$$

Proof. Unwrap the definitions and invoke the chain rule for vector functions. $\square$

Important Remark. Here is a good way to think of the chain rule. Given an coordinate $\mathbf{x}$ on S, for each $j = 1 \ldots, n$ we let

$$\frac{\partial}{\partial x_j}$$

be the function which assigns $\frac{\partial y}{\partial x_j}$ to the variable y on S. That is, $\frac{\partial}{\partial x_j}$ is an *operation* you apply to one variable on S to get another variable on S, or at least a subset of S. The above formula amounts to the statement if $\mathbf{t}$ is another coordinate on S then

$$\frac{\partial}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial t_j} \frac{\partial}{\partial x_i}.$$

Example. Polar coordinates. Let $S = \mathbf{R}^2 \sim \{(a, b) \in \mathbf{R}^2 : a \leq 0, \text{ and } b = 0\}$. Define real valued functions x, y, r, θ on S by setting

$$x(a, b) = a \qquad \text{and} \qquad y(a, b) = y;$$

requiring that the range of r equal $(0, \infty)$; that the range of θ equal $(-\pi, \pi$; and requiring that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Then

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta};\\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial r} + \sin\theta\frac{\partial}{\partial \theta};\\
\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.
\end{aligned}$$

Exercise. Obtain the formula for the Laplacian in polar coordinates:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}.$$

Exercise. Do the analogue of the above for cylindrical coordinates.

Exercise. Do the analogue of the above for spherical coordinates.