1. Integrals in polar coordinates.

We let

$$P : \mathbb{R}^2 \to \mathbb{R}^2$$

be defined by

$$P(r, \theta) = (r \cos\theta, r \sin\theta) \quad \text{for } (r, \theta) \in \mathbb{R}^2.$$

Suppose $0 \le R_0 < R_1 < \infty$ and $0 \le \Theta_0 < \Theta_1 \le 2\pi$ and let

$$A = \{(r, \theta) : R_0 \le r \le R_1 \text{ and } \Theta_0 \le \theta \le \Theta_1\}.$$

Suppose

$$f : P[A] \to \mathbb{R}$$

and f is Riemann integrable over $P[A]$.

Suppose M, N are positive integers,

$$R_0 = r_0 \le r_1 \le \cdots \le r_M = R_1 \quad \text{and} \quad \Theta_0 = \theta_0 \le \theta_1 \le \cdots \le \theta_N = \Theta_1.$$

Suppose as well that

$$r_{i-1} \le \rho_i \le r_i, \quad i = 1, \ldots, M, \quad \text{and} \quad \theta_{j-1} \le \phi_j \le \theta_j, \quad j = 1, \ldots, N.$$

Let

$$S = \sum_{i=1}^{M} \sum_{j=1}^{N} f \circ P(\rho_i, \phi_j) \rho_i (r_i - r_{i-1})(\theta_j - \theta_{j-1}) = \sum_{i=1}^{M} \sum_{j=1}^{N} f \circ P(\rho_i, \phi_j) \rho_i \text{Area}(B_{i,j})$$

where we have set

$$B_{i,j} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j] \quad \text{for } i = 1, \ldots, M \text{ and } j = 1, \ldots, N.$$

Then S is a Riemann sum for

$$\int\int_A f(r \cos\theta, r \sin\theta) r \, dr d\theta$$

provided $A \ni (r, \theta) \mapsto f(r \cos\theta, r \sin\theta) r$ is Riemann integrable. We now show that this is indeed the case and that

$$\int\int_{P[A]} f(x, y) \, dx dy = \int\int_A f(r \cos\theta, r \sin\theta) r, dr d\theta. \tag{1}$$

Let

$$T = \sum_{i=1}^{M} \sum_{j=1}^{N} f(P(\rho_i, \theta_j)) \text{Area}(P[B_{i,j}]).$$

It is a relative simple matter to show that as the diameter of the sets $P[B_{i,j}]$ tends to zero that T approaches

$$\int\int_{P[A]} f(x, y) \, dx dy.$$

A simple calculation yields

$$\text{Area}(P[B_{i,j}]) = \left(r_{i-1} + \frac{1}{2}(r_i - r_{i-1}) \right) \text{Area}(B_{i,j})$$

so that

$$S - T = \sum_{i=1}^{M} \sum_{j=1}^{N} f(P(\rho_i, \theta_j)) \big((\rho_i - r_{i-1}) + \frac{1}{2}(r_i - r_{i-1}) \big) \text{Area}(B_{i,j})$$

the absolute value of which does not exceed

$$\frac{3}{2}M \max_{i=1,\ldots,M}(r_i - r_{i-1})\text{Area}(A)$$

where M is an upper bound for $|f|$ on $P[A]$. It follows that $f \circ P$ is Riemann integrable over A and (1) holds.

Definition 1.1. Suppose X and Y are sets, $f : X \to Y$ and $A \subset X$. We say f **is one-one on** A if

$$a_1 \in A,\ a_2 \in A \text{ and } f(a_1) = f(a_2) \ \Rightarrow\ f(a_1) = f(a_2).$$

By a relatively simple approximation argument one use (1) to prove the following theorem.

Theorem 1.1. Suppose $A \subset \mathbb{R}^2$ and P restricted to A is one-one. Suppose f is Riemann integrable over $P[A]$. The $f \circ P$ is Riemann integrable over A and

$$\int\int_{P[A]} f(x,y)\,dxdy = \int\int_A f(r\cos\theta, r\sin\theta)r, drd\theta.$$