Math 103.02 Quiz Ten

Due Monday, November 29.

I have neither given nor received aid in the completion of this test.
Signature:

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(1) Let C be the segment joining $(1,1)$ to $(3,5)$. Calculate

$$\int_C x\,dx \quad \int_C x\,ds \quad \text{and} \quad \int_C (x,y) \bullet \mathbf{T}\,ds.$$

Solution. Let $P(t) = (x(t), y(t)) = (1+2t, 1+4t)$, $0 \le t \le 1$. Then P parameterizes C and $P'(t) = (2,4)$, $0 \le t \le 1$, so

$$\int_C x\,dx = \int_0^1 (1+2t)\,2dt = 4 \quad (\text{since } dx = 2\,dt);$$

$$\int_C x\,ds = \int_0^1 (1+2t)\sqrt{2^2+4^2}\,dt = 4\sqrt{5};$$

$$\int_C (x,y) \bullet \mathbf{T}\,ds = \int_0^1 (1+2t, 1+4t) \bullet (2,4)\,dt = 16.$$

A better way to do the first of these is to recognize that $(x,0) = \nabla f(x,y)$ where $f(x,y) = x^2/2$ for $(x,y) \in \mathbb{R}^2$ so

$$\int_C x\,dx = \int \nabla f \bullet \mathbf{T}\,dx = f(3,5) - f(1,1) = \frac{3^2}{2} - \frac{1^2}{2} = 4.$$

You *cannot* do the second one this way. A better way to do the third of these is to recognize that $(x,y) = \nabla f(x,y)$ where $f(x,y) = (x^2+y^2)/2$ for $(x,y) \in \mathbb{R}^2$ so

$$\int_C (x,y) \bullet \mathbf{T}\,ds = f(3,5) - f(1,1) = \frac{3^2+5^2}{2} - \frac{1^2+1^2}{2} = 16.$$

(2) Let C be curve joining $(1,0,0)$ to $(-1,0,0)$ which lies in

$$\left\{(x,y,z) : x^2 + \frac{y^2}{2} + \frac{z^2}{2} = 1,\ y = z \text{ and } z \ge 0\right\}.$$

Calculate

$$\int_C x\,ds \quad \text{and} \quad \int_C (x,0,z) \bullet \mathbf{T}\,ds.$$

Solution. The projection of this curve on the xy-plane equals $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ and } y \ge 0\}$; (note that since $z \ge 0$ and $y = z$ on the curve we have

$y \geq 0$ on the curve). So

$$(x(t), y(t), z(t)) = (\cos t, \sin t, \sin t), \quad 0 \leq t \leq \pi,$$

is a parameterization of the curve. Thus

$$\int_C x\,ds = \int_0^\pi \cos t \sqrt{(-\sin t)^2 + (\cos t)^2 + (\cos t)^2}\,dt = 0,$$

which can also be seen by the fact that the curve is symmetric with respect to reflection across the plane $x = 0$.

Finally,

$$\begin{aligned}
&\int_C (x, 0, z) \bullet \mathbf{T}\,ds \\
&\qquad = \int_0^\pi (\cos t, 0, \sin t) \bullet (-\sin t, \cos t, \cos t)\,dt \\
&\qquad = \int_0^\pi (\cos t)(-\sin t) + (\sin t)(\cos t)\,dt \\
&\qquad = \int_0^\pi 0\,dt = 0.
\end{aligned}$$