Math 103.02 Quiz Nine

I have neither given nor received aid in the completion of this test.
Signature:

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Let T be the solid in $\mathbb{R}^3$ consisting of the points (x, y, z) such that

$$1 \le x^2 + y^2 + z^2 \le 9 \quad \text{and} \quad z^2 \le x^2 + y^2.$$

Use spherical coordinates to compute the volume of T. (Hint: The solid you integrate over in (ρ, ϕ, θ) space is a box.)

Solution. Let

$$S(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \quad \text{for } (\rho, \phi, \theta) \in \mathbb{R}^3.$$

If $(x, y, z) = S(\rho, \phi, \theta)$ then

$$z^2 \le x^2 + y^2 \Leftrightarrow \; \rho^2 \cos^2\phi \le \rho^2 \sin^2\theta \;\Leftrightarrow\; |\cos\phi| \le |\sin\phi|.$$

If follows that if

$$A = \{(\rho, \phi, \theta) \in \mathbb{R}^3 : 0 < \rho < \infty \text{ and } \pi/4 < \phi < 3\pi/4\}$$

then S carries A in one-one fashion onto

$$Q = \{(x, y, z) : 0 < z^2 < x^2 + y^2\}.$$

Also, it is clear that S carries

$$B = \{(\rho, \phi, \theta) \in \mathbb{R}^3 : 1 < \rho < 3\}$$

in one-one fashion onto

$$R = \{(x, y, z) : 1^2 < x^2 + y^2 + z^2 < 3^2\}.$$

Thus S carries $A \cap B$ in one-one fashion onto $Q \cap R$. It follows that

$$\begin{aligned}
\int\int\int_T 1 \, dxdydz &= \int\int\int_{Q\cap R} 1 \, dxdydz \\
&= \int\int\int_{A\cap B} \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \int_0^{2\pi} \left(\int_{\pi/4}^{3\pi/4} \left(\int_1^3 \rho^2 \sin\phi \, d\rho \right) d\phi \right) d\theta \\
&= \frac{52\pi\sqrt{2}}{3}.
\end{aligned}$$