Math 103.02; Fall 2010; Test One

I have neither given nor received aid in the completion of this test.
Signature:

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TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

		Your Score
1	5 pts.	
2	20 pts.	
3	5 pts.	
4	5 pts.	
5	10 pts.	
6	15 pts.	
7	10 pts.	
8	5 pts.	
9	5 pts.	
10	10 pts.	
11	15 pts.	
Total	105 pts.	

The average on this 50 minute test was 54.16 and the standard deviation was 17.25.

1. 5 pts. Let P be the plane consisting of those (x, y, z) such that $x + y + z = 1$. Exhibit parametric equations for the line passing through $(1, 2, 3)$ which meets P in a right angle.

Solution. Let $\mathbf{n} = (1, 1, 1)$; then $\mathbf{n}$ is normal to P so

$$\mathbf{r}(t) = (1, 2, 3) + t(1, 1, 1), \quad t \in \mathbb{R},$$

parameterizes the line.

2. (a) **5 pts.** Show that the points $(1, 1, 0), (1, 0, 0), (0, 1, 1)$ do not lie on a line.

Solution. Let $\mathbf{v} = (1, 0, 0) - (1, 1, 0) = (0, -1, 0)$ and let $\mathbf{w} = (0, 1, 1) - (1, 1, 0) = (-1, 0, 1)$. Then

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = -\mathbf{i} - \mathbf{k} = (-1, 0, -1).$$

Since $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$ the points do not lie on a line.

(b) **5 pts.** Let P be the plane containing the points in part (a). Exhibit scalars a, b, c, d such the point (x, y, z) is in P if and only if

$$ax + by + cz = d.$$

Solution. $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is normal to P and $(1, 1, 0)$ lies in P. So

$$-x - z = (x, y, z) \bullet (-1, 0, -1) = (1, 1, 0) \bullet (-1, 0, -1) = -1$$

is an equation for P so we may take

$$a = -1, \quad b = 0, \quad c = -1, \quad d = -1.$$

(c) **10 pts.** Let $\mathbf{q} = (1, 2, 3)$. Show that $\mathbf{q}$ does not lie on P and determine the point $\mathbf{p}$ in P which is closest to $\mathbf{q}$.

Solution. $(-1)1 + (0)2 + (-1)3 = -4 \neq -1$ so $\mathbf{q}$ does not lie in P. Moreover,

$$\mathbf{r}(t) = \mathbf{q} + t\mathbf{n} = (1, 2, 3) + t(-1, 0, -1), \quad t \in \mathbb{R}$$

parameterizes the line passing through $\mathbf{q}$ perpendicular to P so $\mathbf{p} = \mathbf{r}(t)$ if

$$-1 = \mathbf{r}(t) \bullet \mathbf{n} = ((1, 2, 3) + t(-1, 0, -1)) \bullet (-1, 0, -1) = -4 + 2t$$

so $t = 3/2$ and

$$\mathbf{p} = \mathbf{r}\left(\frac{3}{2}\right) = (1, 2, 3) + \frac{3}{2}(-1, 0, -1) = \frac{1}{2}(-1, 4, 3).$$

3. 5 pts. Suppose for P_i is the plane with equation

$$a_i x + b_i y + c_i z = d_i$$

for each $i = 1, 2$. How do you tell if P_1 is parallel to P_2?

Solution. Let $\mathbf{n}_i = (a_i, b_i, c_i)$ for $i = 1, 2$. Then $\mathbf{n}_i$ is normal to P_i, $i = 1, 2$, so $P_1 \parallel P_2$ if and only if $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$.

4. 5 pts. Compute $\mathbf{comp}_{(3,0,4)}(1, 2, 3)$.

Solution. We have

$$\mathbf{comp}_{(3,0,4)}(1, 2, 3) = \frac{(1, 2, 3) \bullet (3, 0, 4)}{|(3, 0, 4)|} = \frac{1 \cdot 3 + 2 \cdot 0 + 3 \cdot 4}{\sqrt{3^2 + 0^2 + 4^2}} = 3.$$

5. 10 pts. Suppose I is an open interval and $\mathbf{r} : I \to \mathbb{R}^3$ is a twice continuously differentiable curve in $\mathbb{R}^3$. Suppose $t_0 \in I$ and

$$\mathbf{r}'(t_0) = (1, 0, 1), \quad \mathbf{r}''(t_0) = (1, 1, 1).$$

Determine

$$\mathbf{T}(t_0), \quad |\mathbf{v}|'(t_0), \quad \mathbf{N}(t_0), \quad \kappa(t_0).$$

Solution. We have

$$\mathbf{v}(t_0) = \mathbf{r}'(t_0) = (1, 0, 1) \quad \text{so} \quad |\mathbf{v}|(t_0) = \sqrt{2}$$

and

$$\mathbf{T}(t_0) = (\mathbf{v}/|\mathbf{v}|)(t_0) = \frac{1}{\sqrt{2}}(1,0,1).$$

We have

$$\mathbf{a}(t_0) = \mathbf{r}''(t_0) = (1,1,1)$$

and

$$|\mathbf{v}|'(t_0) = (\mathbf{a} \bullet \mathbf{T})(t_0) = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

We have

$$(\kappa|\mathbf{v}|^2\mathbf{N})(t_0) = (\mathbf{a} - (\mathbf{a} \bullet \mathbf{T})\mathbf{T})(t_0) = (1,1,1) - \sqrt{2}\frac{1}{\sqrt{2}}(1,0,1) = (0,1,0).$$

Since $\kappa(t_0) > 0$ we find that

$$\mathbf{N}(t_0) = (0,1,0) \quad \text{and} \quad \kappa(t_0) = \frac{1}{|\mathbf{v}|^2(t_0)} = \frac{1}{2}.$$

6. 15 pts. Let $f(x,y) = xy - x + 2$ and let $R = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 4 - x^2\}$. Find the maxima and minima of f on R. (Note that f is continuous and R is closed and bounded so both maxima and minima exist.)

Solution. First we find the critical points. We have

$$\frac{\partial f}{\partial x} = y - 1 \quad \text{and} \quad \frac{\partial f}{\partial y} = x$$

so the unique critical point is $(0,1)$ which lies in R.

Next we note the boundary of R consists of the arcs which are the ranges of the curves

$$C_1(x) = (x,0) \quad \text{for } -2 < x < 2$$

and

$$C_2(x) = (x, 4 - x^2) \quad \text{for } -2 < x < 2$$

together with the endpoints of these arcs, namely the points $(\pm 2, 0)$. Since

$$\frac{d}{dx}f(C_1(x)) = \frac{d}{dx}(-x+2) = -1$$

there are no candidates for minimum or maximum points of f on R on C_1. Since

$$\frac{d}{dx}f(C_2(x)) = \frac{d}{dx}(x(4-x^2) - x + 2) = 3(1-x^2)$$

we find that $(\pm 1, 3)$ are candidates for minimum or maximum points of f on R on C_2. Finally, we consider the endpoints $(\pm 2, 0)$.

We consider the table

x	y	f(x,y)
0	1	2
-1	3	0
1	3	4
2	0	0
-2	0	4

Thus the minimum value of f on R is 0 which occurs at $(-1,3)$ and $(2,0)$ and the maximum value of f on R is 4 which occurs at $(1,3)$ and $(-2,0$.

7. 10 pts. Let

$$f(x,y) = \begin{cases} \frac{-x}{x^2+y^2} & \text{if } x < y, \\ \frac{y}{x^2+y^2} & \text{if } x \geq y. \end{cases}$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, what is it? Why?

Solution. If $0 < x < \infty$ we have $f(x,0) = 0$ and $f(x,x) = 1/x$. Thus

$$\lim_{x\downarrow 0} f(x,0) = 0 \quad \text{and} \quad \lim_{x\downarrow 0} f(x,x) = \infty$$

so the limit does not exist.

8. 5 pts. Exhibit an equation for the tangent plane to the graph of $z = \cos xy$ at $(1, \pi/2, 0)$.

Solution. Note that $f(1,\pi/2) = 0$; otherwise the problem is incorrectly posed. We have

$$\frac{\partial f}{\partial x} = -y \sin xy \quad \text{which at } (1,\pi/2) \text{ equals } -\tfrac{\pi}{2}$$

and

$$\frac{\partial f}{\partial y} = -x \sin xy \quad \text{which at } (1,\pi/2) \text{ equals } -1.$$

Thus

$$z = -\frac{\pi}{2}(x-1) + (-1)\left(y - \frac{\pi}{2}\right) = -\frac{\pi}{2}x - y + \pi$$

is the desired equation.

9. 5 pts. Calculate

$$\frac{\partial^2}{\partial x \partial y} e^{xyz}.$$

Solution. We have

$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} e^{xyz} &= \frac{\partial}{\partial x}\left(xze^{xyz}\right) \\ &= ze^{xyz} + (xz)(yz)e^{xyz} \\ &= z(1+xyz)e^{xyz}. \end{aligned}$$

9. 11 pts. Suppose g, h are continuously differentiable functions on the interval I, f is a continuously differentiable function on $\mathbb{R}^2$ and

$$w(t) = f(g(t), h(t)) \quad \text{for } t \in I.$$

Suppose

$$2 \in I, \quad g(2) = 1, \quad g'(2) = 1, \quad h(2) = 4, \quad h'(2) = 3$$

as well as

$$f(1,4) = 5, \quad \frac{\partial f}{\partial x}(1,4) = 3$$

and

$$w'(2) = 4.$$

Determine

$$\frac{\partial f}{\partial y}(1,4).$$

Solution. The Chain Rule says that

$$w'(t) = \frac{\partial f}{\partial x}(g(t),h(t))g'(t) + \frac{\partial f}{\partial y}(g(t),h(t))h'(t)$$

for any $t \in I$. If $t = 2$ then $(g(t),h(t)) = (1,4)$ so

$$4 = \frac{\partial f}{\partial x}(1,4))(1) + \frac{\partial f}{\partial y}(1,4)(3) = (3)(1) + \frac{\partial f}{\partial y}(1,4)(3)$$

so

$$\frac{\partial f}{\partial y}(1,4) = \frac{1}{3}(4-3) = \frac{1}{3}.$$

11. 15 pts. Suppose $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ are points in $\mathbb{R}^3$ which do not lie in a plane. Let P be the plane passing through $\mathbf{a},\mathbf{b},\mathbf{c}$ and let Q be the plane passing through the midpoints of the segments joining $\mathbf{a}$ to $\mathbf{d}$, $\mathbf{b}$ to $\mathbf{d}$ and $\mathbf{c}$ to $\mathbf{d}$, respectively. Show that P and Q are parallel.

Solution. The vector

$$(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})$$

is normal to P. The vectors

$$\mathbf{m_a} = \frac{1}{2}(\mathbf{d}-\mathbf{a}), \quad \mathbf{m_b} = \frac{1}{2}(\mathbf{d}-\mathbf{b}), \quad \mathbf{m_c} = \frac{1}{2}(\mathbf{d}-\mathbf{c})$$

are the midpoints of the segments joining $\mathbf{a}$ to $\mathbf{d}$, $\mathbf{b}$ to $\mathbf{d}$ and $\mathbf{c}$ to $\mathbf{d}$, respectively. We have

$$(\mathbf{m_b}-\mathbf{m_a}) \times (\mathbf{m_c}-\mathbf{m_a}) = \left(\frac{1}{2}(\mathbf{a}-\mathbf{b})\right) \times \left(\frac{1}{2}(\mathbf{a}-\mathbf{c})\right) = \frac{1}{4}((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}))$$

from which we infer that P and Q are parallel.