Math 103.02; Fall 2010; Test Two

I have neither given nor received aid in the completion of this test.
Signature:

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TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

		Your Score
1	10 pts.	
2	5 pts.	
3	10 pts.	
4	5 pts.	
5	10 pts.	
6	15 pts.	
7	10 pts.	
8	15 pts.	
9	10 pts.	
10	15 pts.	
Total	105 pts.	

The average was 75.93 and the standard deviation was 17.52.

1. 10 pts. Fix $a, b \in \mathbb{R}$ and let $f(x, y) = x^2 - 4xy + y^2 + ax + by$ for $(x, y) \in \mathbb{R}^2$. Use the the second derivative test to show that f has no maxima or minima.

Solution. We have

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} (x, y) = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$$

the determinant of which is -12 which is negative. Therefore any critical point (there is only one but we don't need to know that) is a saddle point. So, since any minimum or maximum is a critical point, there can be neither.

2. 5pts. Let $f(x, y, z) = xy^2z^3$. Calculate $\nabla f(1, 1, 1)$. Find an equation for the tangent plane to the surface $xy^2z^3 = 1$ at $(1, 1, 1)$.

Solution. We have

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial x}(x, y, z) \right) = (y^2z^3, 2xyz^3, 3xy^2z^2)$$

which at $(1,1,1)$ is $(1,2,3)$. An equation for tangent plane to the surface is

$$((x,y,z)-(1,1,1)) \bullet \nabla f(1,1,1) = 0$$

or

$$x + 2y + 3z = 6.$$

3. 10 pts. Let $S = \{(x,y,z) : x^2 + 2y^2 + 3z^2 = 12\}$ and let $f(x,y,z) = x + 2y$ for $(x,y,z) \in \mathbb{R}^3$. Use the method of Lagrange multipliers to find the maxima and minima of f on S.

Solution. We calculate

$$(\nabla f \times \nabla g)(x,y,z) = (1,2,0) \times (2x,4y,6z) = (12z, 6z, 4y-4x)$$

which is $(0,0,0)$ if and only if $z = 0$ and $y = x$. Substituting $(x,x,0)$ into the constraint $x^2+2y^2+6z^2 = 12$ we find that $3x^2 = 12$ or $x = \pm 2$. Since $f(2,2,0) = 6$ and $f(-2,-2,0) = -6$ we find that $(2,2,0)$ is the unique maximum and $(-2,-2,0)$ is the unique minimum.

4. 5 pts. Let $R = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le x^2\}$. Calculate

$$\int\int_R x + 2y^2 \, dxdy.$$

Solution.

$$\begin{aligned}\int_0^1 \left(\int_0^{x^2} x + 2y^2 \, dy \right) dx &= \int_0^1 \left(xy + \frac{2y^3}{3} \right)_{y=0}^{y=x^2} \\ &= \int_0^1 x^3 + \frac{2x^6}{3} \, dx \\ &= \frac{29}{84}.\end{aligned}$$

5. 10 pts. Calculate $\int\int_R f(x,y)\, dxdy$ where R is the region bounded by the parabolas $x = 1 - y^2$ and $x = y^2 - 1$ and $f(x,y) = y^2$ for $(x,y) \in \mathbb{R}^2$.

Solution. The parabolas intersect at $(1,1)$ and $(1,-1)$ so the the projection of R on the y-axis is $[-1,1]$ on which $y^2 - 1 \le 1 - y^2$. Consequently

$$\int\int_R y^2 \, dxdy = \int_{-1}^1 \left(\int_{y^2-1}^{1-y^2} y^2 \, dx \right) dy = \frac{8}{15}.$$

6. 15 pts. Let

$$T = \{(x,y,z) : 0 \le x^2 \le y \le z \le 1\}.$$

Calculate the volume of T.

Solution One. Slicing. For each $z \in \mathbb{R}$ let $S_z = \{(x,y) \in \mathbb{R}^2 : (x,y,z) \in T\}$. Then $S_z = \emptyset$ if $|z| > 1$ and $S_z = \{(x,y) \in \mathbb{R}^2 : 0 \le x^2 \le y \le z\}$ so

$$\text{Area}(S_z) = \int_{-\sqrt{z}}^{\sqrt{z}} z - x^2 \, dx = \frac{4}{3} z^{3/2}.$$

It follows that

$$\text{Volume}(T) = \int_0^1 \text{Area}(S_z)\,dz = \int_0^1 \frac{4}{3} z^{3/2}\,dz = \frac{8}{15}.$$

Solution Two. Solid between two graphs. Let $R = \{(x,y) \in \mathbb{R}^2 : 0 \le x^2 \le y \le 1\}$. Then $T = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R \text{ and } y \le z \le 1\}$ and $y \le 1$ for $(x,y) \in R$. Thus

$$\text{Volume}(T) = \int\int_R 1 - y\,dxdy = \int_{-1}^1 \left(\int_{x^2}^1 1 - y\,dy \right) dx = \frac{8}{15}.$$

7. 10 pts. Use polar coordinates to calculate

$$\int_0^1 \left(\int_0^{\sqrt{1-y^2}} x^2 + y^2\,dx \right) dy.$$

Solution. Let $R = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1 \text{ and } 0 \le x \le \sqrt{1-y^2}\}$ and note that the polar coordinate transformation maps the interior of $Q = \{(r,\theta) : 0 \le r \le 1 \text{ and } 0 \le \theta \le \pi/2\}$ in one-one fashion onto the interior of R. Consequently,

$$\begin{aligned}
&\int_0^1 \left(\int_0^{\sqrt{1-y^2}} x^2 + y^2\,dx \right) dy \\
&\quad = \int\int_R x^2 + y^2\,dxdy \\
&\quad = \int\int_Q \left((r\cos\theta)^2 + (r\sin\theta)^2 \right) r\,drd\theta \\
&\quad = \int_0^1 \left(\int_0^{\pi/2} r^3\,dr \right) d\theta \\
&\quad = \frac{\pi}{8}.
\end{aligned}$$

8. 15 pts. Calculate

$$\int\int\int_T x\,dxdydz$$

where T is the solid bounded by $y = z^2$, $z = y^2$, $x + y + z = 2$ and $x = 0$.

Solution. Let $R = \{(y,z) : y \le z^2 \text{ and } z < y^2\}$. Note that

$$R = \{(y,z) : 0 < y < 1 \text{ and } y^2 \le z \le \sqrt{y}\}.$$

If $(y,z) \in R$ then $2-y-z \geq 0$ since $0 < y < 1$ and $0 < z < 1$. Thus $T = \{(x,y,z) : (y,z) \in R \text{ and } 0 \leq x \leq 2-y-z\}$ and

$$\begin{aligned}
\int\int\int_T x\,dxdydz &= \int_R \left(\int_0^{2-y-z} x\,dx\right) dydz \\
&= \int_0^1 \left(\int_{y^2}^{\sqrt{y}} \left(\int_0^{2-y-z} x\,dx\right) dz\right) dy \\
&= \frac{33}{140}.
\end{aligned}$$

9. 10 pts. Use spherical coordinates to find the z coordinate of the centroid of

$$T = \{(x,y,z) : x^2+y^2+z^2 \leq 1 \text{ and } 0 \leq z\}.$$

The volume of T is $2\pi/3$.

Solution. Let $\overline{z}$ be the z-coordinate of the centroid. Let

$$B = \{(\rho,\phi,\theta) : 0 \leq \rho \leq 1, 0 \leq \phi \leq \pi/2 \text{ and } 0 \leq \theta \leq 2\pi\}.$$

Since the $S(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)$ carries the interior of B in one-one fashion onto T, we have

$$\begin{aligned}
\overline{z} &= \frac{1}{\text{Volume}T} \int\int\int_T z\,dxdydz \\
&= \frac{3}{2\pi} \int\int\int_B \rho\cos\phi\,\rho^2\sin\phi\,d\rho d\phi d\theta \\
&= \frac{3}{2\pi} \int_0^{2\pi} \left(\int_0^{\pi/2} \left(\int_0^1 \rho^3 \sin\phi\cos\phi\,d\rho\right) d\phi\right) d\theta \\
&= \frac{3}{8}.
\end{aligned}$$

15. 10pts. Let S be the set of points in $\mathbb{R}^3$ whose distance to the origin is $\sqrt{2}$ and whose z-coordinate is greater than 1. Calculate the surface area of S.

Solution. In spherical coordinate (ρ,ϕ,θ), $z > 1$ and $\rho = \sqrt{2}$ amount to $1 < z = \rho\cos\phi\sqrt{2}$ or $\cos\phi > 1/\sqrt{2}$ which holds if $0 \leq \phi < \pi/4$. So if

$$R = \{(\phi,\theta) : 0 \leq \phi \leq \pi/4 \text{ and } 0 \leq \theta \leq \sqrt{2}\}$$

and

$$P(\rho,\theta) = \sqrt{2}(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

for $(\rho,\theta) \in R$ then P parameterizes S. Moreover, from class or the book, we have

$$\left|\frac{\partial P}{\partial\phi} \times \frac{\partial P}{\partial\theta}\right|(\rho,\theta) = (\sqrt{2})^2\sin\phi = 2\sin\phi$$

for $(\phi, \theta) \in R$. Thus

$$\begin{aligned}\text{Area}(S) &= \int\int_R 2\sin\phi\, d\rho d\theta \\ &= 2\int_0^{2\pi}\left(\int_0^{\pi/4}\sin\phi\, d\phi\right)d\theta \\ &= 2\pi(2-\sqrt{2}).\end{aligned}$$