Math 103.02; Fall 2010; Test Three

I have neither given nor received aid in the completion of this test.
Signature:

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

		Your Score
1	20 pts.	
2	15 pts.	
3	10 pts.	
4	10 pts.	
5	10 pts.	
6	10 pts.	
7	15 pts.	
8	15 pts.	
Total	105 pts.	

The average was 58.5 and the standard deviation was 21.8.

1. Let

$$B = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ 1 < x^2 - y^2 < 4 \text{ and } 4 < x^2 + y^2 < 9\}$$

and let

$$F(x, y) = (x^2 - y^2, x^2 + y^2) \quad \text{for } (x, y) \in B.$$

I tell you that F carries A in one-to-one fashion onto the rectangle

$$\{(u, v) \in \mathbb{R}^2 : 1 < u < 4 \text{ and } 4 < v < 9\}.$$

(a) **10 pts.** Use the Change of Variables formula to calculate

$$\int\int_B x^2 + y^2\, dxdy.$$

(b) **10 pts.** Calculate the inverse of F.

Solution. First note that

$$J_F(x, y) = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 8xy \neq 0 \quad \text{for } (x, y) \in B.$$

Let A be the rectangle

$$\{(u,v) \in \mathbb{R}^2 : 1 < u < 4 \text{ and } 4 < v < 9\}.$$

Then F^{-1} carries A in one-to-one fashion onto B since F carries B in one-to-one fashion onto A. Moreover, by the Chain Rule and the product rule for determinants (see (9) on page 1005) we have

$$J_{F^{-1}}(u,v) = \frac{1}{J_F(F^{-1}(u,v))} \quad \text{for } (u,v) \in A.$$

By the Change of Variable Formula for Multiple Integrals (Theorem 1 on page 1004) we have

$$\begin{aligned} &\int\int_B f(x,y)\,dxdy \\ &\qquad = \int\int_{F^{-1}[A]} f(x,y)\,dxdy \\ &\qquad = \int\int_A f(F^{-1}(u,v))J_{F^{-1}}(u,v)\,dudv \\ &\qquad = \int\int_A \frac{f(F^{-1}(u,v))}{J_F(F^{-1}(u,v))}\,dudv. \end{aligned} \tag{1}$$

In the original statement of the problem I should have had you integrate $f(x,y) = xy$ instead of $f(x,y) = x^2 + y^2$ because this leads to an easy integral as follows. If $f(x,y) = xy$ for $(x,y) \in B$ we have

$$\frac{f(x,y)}{J_F(x,y)} = \frac{xy}{8xy} = \frac{1}{8}$$

so

$$\int\int_B xy\,dxdy = \int\int_A \frac{1}{8}\,dudv = \frac{(4-1)(9-4)}{8} = \frac{15}{8}.$$

Back to the problem as stated. Now let $f(x,y) = x^2 + y^2$ for $(x,y) \in B$. Fix $(x,y) \in B$ and let

$$(u,v) = F(x,y) = (x^2 - y^2, x^2 + y^2).$$

It follows that

$$F^{-1}(u,v) = (x,y).$$

Now

$$x^2y^2 = \frac{(x^2+y^2) - (x^2-y^2)}{4} = \frac{v-u}{4}$$

so

$$xy = \frac{\sqrt{v-u}}{2} \quad \text{since } v > u.$$

Consequently,

$$\frac{f(F^{-1}(u,v))}{J_F(F^{-1}(u,v))} = \frac{v}{4\sqrt{v-u}}$$

which gives

$$\begin{aligned}
&\int\int_B x^2 + y^2\,dxdy \\
&\quad = \int\int_A \frac{v}{4\sqrt{v-u}}\,dudv \\
&\quad = \int_1^4 \left(\int_4^9 \frac{v}{4\sqrt{v-u}}\,dv\right) du \\
&\quad = \frac{464}{15}\sqrt{2} - \frac{14}{5}\sqrt{3} - \frac{35}{3}\sqrt{5}
\end{aligned}$$

(I meant to have you end up with a real easy integral. So I should have told you to integrate xy instead of $x^2 + y^2$.)

To calculate the inverse of F we need to solve $u = x^2 - y^2$ and $v = x^2 + y^2$ for u and v. *Keep in mind that $v > u$ for $(x, y) \in B$.* Adding these equations we get $u + v = 2x^2$ so $x = \sqrt{(u+v)/2}$ and subtracting the second from the first we get $v - u = 2y^2$ so $y = \sqrt{(v-u)/2}$. That is,

$$F^{-1}(u, v) = \left(\sqrt{\frac{u+v}{2}}, \sqrt{\frac{v-u}{2}}\right) \quad \text{for } (u, v) \in A.$$

Note that having these formulae will allow you to calculate $J_{F^{-1}}(u, v)$ directly, giving another way to do (a).

2. Let

$$P(x, y) = 2x^2 + x \quad \text{and} \quad Q(x, y) = -3x^2 + y \quad \text{for } (x, y) \in \mathbb{R}^2$$

and let R be the triangle in $\mathbb{R}^2$ with vertices $(0, 0), (1, 1), (0, 1)$.

(a) **5 pts.** Calculate

$$\int\int_R x\,dxdy.$$

(b) **5 pts.** Calculate

$$\int_C P\,dx + Q\,dy$$

where C is the boundary of R traversed in the counterclockwise sense.

(c) **5 pts.** Explain how the answer to either one of (a) or (b) may be used to find the answer to the other.

Solution. We have

$$\int\int_R x\,dxdy = \int_0^1 \left(\int_0^x x\,dx\right) dy = \frac{1}{6}.$$

Moreover, if C_i, $i = 1, 2, 3$, are the segments joining $(0,0)$ to $(1,1)$; $(1,1)$ to $(0,1)$; and $(0,1)$ to $(0,0)$, respectively, we have

$$\begin{aligned}
&\int_C P\,dx + Q\,dy \\
&= \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) P\,dx + Q\,dy \\
&= \int_0^1 (2t^2 + t)\,dt + (-3t^2 + t)\,dt \\
&\qquad - \int_0^1 (2t^2 + t)\,dt + (-3t^2 + 1)\,dt \\
&\qquad - \int_0^1 (20^2 + 0)\,dt + (-30^2 + t)\,dt \\
&= -1.
\end{aligned}$$

Now Green's Theorem says

$$\int_C P\,dx + Q\,dy = \int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dxdy = \int\int_R -6x\,dxdy$$

so the answer to (b) should be -6 times the answer to (a).

3. 10 pts. Let

$$\mathbf{F}(x,y) = (y, 0) \quad \text{for } (x,y) \in \mathbb{R}^2$$

and let C be the curve in $\mathbb{R}^3$ which consists of the line segment which goes from $(1,0)$ to $(2,0)$ and then follows the circle with center $(0,0)$ and radius 2 from $(2,0)$ counterclockwise to $(0,2)$. Calculate

$$\int_C \mathbf{F} \bullet \mathbf{T}\,ds.$$

Solution. Let C_1 and C_2 be the line segment which goes from $(1,0)$ to $(2,0)$ and let C_2 be the part of the circle with center $(0,0)$ and radius 2 from $(2,0)$ counterclockwise to $(0,2)$. Then

$$\int_{C_1} \mathbf{F} \bullet \mathbf{T}\,ds = \int_0^1 (0,0) \bullet (1,0)\,dt = 0$$

and

$$\int_{C_2} \mathbf{F} \bullet \mathbf{T}\,ds = \int_0^{\pi/2} (2\sin t, 0) \bullet (-2\sin t, 2\cos t)\,dt = \int_0^{\pi/2} -4\sin^2 t\,dt = -\pi.$$

So

$$\int_C \mathbf{F} \bullet \mathbf{T}\,ds = \int_{C_1} \mathbf{F} \bullet \mathbf{T}\,ds + \int_{C_2} \mathbf{F} \bullet \mathbf{T}\,ds = 0 + (-\pi) = -\pi.$$

4. Let

$$\mathbf{F}(x,y) = (x + y^2, 2xy) \quad \text{for } (x,y) \in \mathbb{R}^2.$$

(a) **5 pts.** Find a continuously differentiable function f on $\mathbb{R}^2$ such that $\mathbf{F} = \nabla f$.

(b) **5 pts.** Calculate

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

where C is a curve which goes from $(1, 1)$ to $(-2, -3)$.

Solution. By partial integration we have

$$f(x, y) = A(y) + \frac{x^2}{2} + xy^2 \quad \text{and} \quad f(x, y) = B(x) + xy^2$$

so we can take $B(x) = x^2/2$ and $A(y) = 0$ and let

$$f(x, y) = \frac{x^2}{2} + xy^2.$$

Because $\mathbf{F} = \nabla f$ we have

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = f(-2, -3) - f(1, 1) = -\frac{35}{2}.$$

5. 10 pts. Let

$$S = \{(x, y, z) : z = xy \ \text{ and } x^2 + y^2 \le 1\}$$

and let

$$\mathbf{F}(x, y, z) = (y, x, 0) \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

Calculate the flux

$$\int\int_S \mathbf{F} \bullet \mathbf{n}\, dS$$

where $\mathbf{n}$ is the upward pointing unit normal to S.

Solution. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ and note that S is the graph of $f(x, y) = xy$ over D. Thus as

$$\mathbf{n}\, dS = (-f_x, -f_y, 1)\, dxdy = (-y, -x, 1)$$

we obtain

$$\begin{aligned}
\int\int_S \mathbf{F} \bullet \mathbf{n}\, dS &= \int\int_D (y, x, 0) \bullet (-y, -x, 1)\, dxdy \\
&= \int\int_D -(x^2 + y^2)\, dxdy \\
&= -\int_0^1 \left(\int_0^{2\pi} (r^2) r\, dr \right) d\theta \\
&= -\frac{\pi}{2}.
\end{aligned}$$

6. 10 pts. Let

$$\mathbf{F}(x, y, z) = (yz, xz, xy) \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

Find a continuously differentiable function f on $\mathbb{R}^3$ such that $\mathbf{F} = \nabla f$.

Solution. By partial integration we obtain

$f(x,y,z) = A(y,z) + xyz, \quad f(x,y,z) = B(x,z) + xyz, \quad f(x,y,z) = C(x,y) + xyz.$

Letting $A(y,z) = 0$, $B(x,z) = 0$ and $C(x,y) = 0$ we obtain

$$f(x,y,z) = xyz \quad \text{for } (x,y,z) \in \mathbb{R}^3.$$

7. Let

$$\mathbf{F}(x,y,z) = (x + e^{yz}, y + \sin xz, z + \cos xy) \quad \text{for } (x,y,z) \in \mathbb{R}^3.$$

(a) **5 pts.** Calculate the divergence of $\mathbf{F}$.

(b) **10 pts.** Use the Divergence Theorem to evaluate

$$\int\int_S \mathbf{F} \bullet \mathbf{n}\, ds$$

where S is the surface which bounds the region

$$T = \{(x,y,z) : z \geq 0 \text{ and } z^2 \leq 25 - x^2 - y^2\}$$

and where $\mathbf{n}$ is the unit normal to S which points out of T. *You only need express your answer as iterated single integrals.*

Solution. We have

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(x + e^{y,z}) + \frac{\partial}{\partial y}(y + \sin xz) + \frac{\partial}{\partial (z} z + \cos xy) = 1 + 1 + 1 = 3.$$

Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25\}$ and note that T is the region whose projection onto the xy-plane is D and which is between $z = 0$ and $z = \sqrt{25 - x^2 - y^2}$. By the Divergence Theorem,

$$\begin{aligned}
&\int\int_S \mathbf{F} \bullet \mathbf{n}\, dS \\
&\quad = \int\int\int_T \nabla \bullet \mathbf{F}\, dV \\
&\quad = 3\text{Volume}(T) \\
&\quad = \int\int_D \sqrt{25 - x^2 - y^2}\, dx dy \\
&\quad = \int_0^5 \left(\int_0^{2\pi} \sqrt{25 - r^2} r\, dr \right) d\theta \\
&\quad = \frac{250\pi}{3}.
\end{aligned}$$

8. Let

$$\mathbf{F}(x,y,z) = (y^2, z^2, x^2) \quad \text{for } (x,y,z) \in \mathbb{R}^3.$$

(a) **5 pts.** Calculate the curl of $\mathbf{F}$.

(b) **10 pts.** Use Stokes's Theorem to evaluate

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

where C is the intersection of the cylinder $x^2 + y^2 = 2y$ with the plane $z = y$ oriented counterclockwise when viewed from above. *You only need to express your answer as iterated single integrals.*

Solution. We have

$$(\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2(z, x, y).$$

Our surface S is the graph of $f(x, y) = y$ over the disk D in $\mathbb{R}^2$ with center $(0, 1)$ and radius 1. Moreover,

$$D = \{(r\cos\theta, r\sin\theta) : 0 \le \theta \le \pi \text{ and } 0 \le r \le 2\sin\theta\}.$$

Thus

$$\mathbf{n}\, dS = (-f_x, -f_y, 1)\, dxdy = (0, -1, 1).$$

Thus

$$\begin{aligned} \int_C \mathbf{F} \bullet \mathbf{T}\, ds &= \int\int_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS \\ &= \int\int_D -2(z, x, y) \bullet (0, -1, 1) \\ &= 2\int\int_D y - x\, dxdy \\ &= 2\int_0^{\pi} \left(\int_0^{2\sin\theta} (r\sin\theta - r\cos\theta) r\, dr \right) d\theta \\ &= 2\pi. \end{aligned}$$