Definition. For $z \in \mathbf{C}$ we *define*

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}); \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}); \quad \cosh z = \frac{1}{2}(e^{z} + e^{-z}); \quad \sinh z = \frac{1}{2}(e^{z} - e^{-z}).$$

Remark In all of what follows we will make use of the fundamental identity

$$(*) \qquad e^{z+w} = e^z e^w, \quad z, w \in \mathbf{C}.$$

Here is a proof of (*) in which we leave out the nontrivial justification of some of the steps. We have

$$\begin{aligned} e^{z+w} &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{z^m}{m!} \frac{w^{n-m}}{(n-m)!} \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} \frac{z^m}{m!} \frac{w^{n-m}}{(n-m)!} \right) \\ &= \sum_{m=0}^{\infty} \left(\frac{z^m}{m!} \sum_{n=m}^{\infty} \frac{w^{n-m}}{(n-m)!} \right) \\ &= \sum_{m=0}^{\infty} \left(\frac{z^m}{m!} \sum_{n=0}^{\infty} \frac{w^{n}}{n!} \right) \\ &= \sum_{m=0}^{\infty} \frac{z^m}{m!} e^w \\ &= e^z e^w. \end{aligned}$$

From the definitions we immediately deduce that

$$(1) \qquad e^{iz} = \cos z + i \sin z$$

and that

$$(2) \qquad \cos^2 z + \sin^2 z = 1, \quad \cosh^2 z - \sinh^2 z = 1.$$

From (*) we obtain the addition formulae

$$(3) \qquad \sin(z+w) = \sin z \cos w + \cos z \sin w, \quad \cos(z+w) = \cos z \cos w - \sin z \sin w.$$

Differentiating term by term the power series that define the exponential function we find that

$$(4) \qquad \frac{d}{dz} e^z = e^z.$$

One could also use (*) to deduce (4). For any constant a we have $\frac{d}{dz} f(az) = a f'(az)$ provided f is differentiable in the complex; this fact, (1) and the definitions immediately imply that

$$(5) \qquad \frac{d}{dz} \cos z = -\sin z; \quad \frac{d}{dz} \sin z = \cos z; \quad \frac{d}{dz} \cosh z = \sinh z; \quad \frac{d}{dz} \sinh z = \cosh z.$$

We have

$$e^{iz}=\sum_{n=0}^{\infty}\frac{(iz)^n}{n!}=\sum_{m=0}^{\infty}\frac{(iz)^{2m}}{(2m)!}+\sum_{m=0}^{\infty}\frac{(iz)^{2m+1}}{(2m+1)!}=\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m}}{(2m)!}+i\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m+1}}{(2m+1)!}. \tag{6}$$

This gives

$$e^{-iz}=\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m}}{(2m)!}-i\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m+1}}{(2m+1)!}. \tag{7}$$

Combining (6) and (7) we get

$$\cos z=\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m}}{(2m!)},\quad \sin z=\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m+1}}{(2m+1)!}. \tag{8}$$

We have

$$e^{z}=\sum_{n=0}^{\infty}\frac{z^n}{n!}=\sum_{m=0}^{\infty}\frac{z^{2m}}{(2m)!}+\sum_{m=0}^{\infty}\frac{z^{2m+1}}{(2m+1)!}. \tag{10}$$

This gives

$$e^{-z}=\sum_{m=0}^{\infty}\frac{z^{2m}}{(2m)!}-\sum_{m=0}^{\infty}\frac{z^{2m+1}}{(2m+1)!}. \tag{11}$$

Combining (10) and (11) we get

$$\cosh z=\sum_{m=0}^{\infty}\frac{z^{2m}}{(2m)!},\quad \sinh z=\sum_{m=0}^{\infty}\frac{z^{2m+1}}{(2m+1)!}. \tag{12}$$

If $z=x+iy$ where x,y are real we have

$$\begin{aligned}\cos z&=\frac{1}{2}\left(e^{i(x+iy)}+e^{-i(x+iy)}\right)\\&=\frac{1}{2}\left(e^{-y+ix}+e^{y-ix}\right)\\&=\frac{1}{2}\left(e^{-y}(\cos x+i\sin x)+e^{y}(\cos x-i\sin x)\right)\\&=\frac{1}{2}\left(\cos x(e^{y}+e^{-y})-i\sin x(e^{y}-e^{-y})\right)\\&=\cos x\cosh y-i\sin x\sinh y.\end{aligned} \tag{13}$$

Changing a sign and dividing by i in the third line of the preceding calculation one obtains

$$\begin{aligned}\sin z&=\frac{1}{2i}\left(e^{-y}(\cos x+i\sin x)-e^{y}(\cos x-i\sin x)\right)\\&=\frac{1}{2i}\left(i\sin x(e^{y}+e^{-y})-\sin x(e^{y}-e^{-y})\right)\\&=\sin x\cosh y+i\cos x\sinh y.\end{aligned} \tag{14}$$

Finally,

$$\cosh z=\cos iz=\cos(-y+ix)=\cos(-y)\cosh x+\sin(-y)\cosh x=\cosh x\cos y-\cosh x\sin y \tag{15}$$

and

(16)

$$\sinh z=-i\sin(iz)=-i\sin(-y+ix)=-i(\sin(-y)\cosh x+i\cos(-y)\sinh x)=\sinh x\cos y+i\cosh x\sin y.$$

Whew!